PLANE TRIGONOMETRY

AND

APPLICATIONS

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PREFACE

The characteristic features of this book may be summarized as follows:

1. *The method of presentation is thoroughly heuristic.* This enables the student to get a firm grasp of the subject by teaching him to recognize the fundamental ideas which underlie and unify the separate steps of the mathematical argument, instead of confusing him by a disconnected dogmatic statement of isolated facts.

2. *The book is divided into two parts.* The first part is devoted to the theoretical and numerical solution of triangles. The second part treats of the functions of the general angle, their addition theorems and other properties, together with applications to simple harmonic curves, simple harmonic and wave motion, and harmonic analysis. Part One is also published separately and is well adapted for use in secondary schools. The complete book is intended for the freshman course in colleges.

3. *The discussion of the solution of triangles,* in Part One, is not interrupted by any digressions about coördinate systems, addition theorems, and the like. It has been thought desirable to postpone to the second part the consideration of all of these matters, which are indeed important but unnecessary for the solution of triangles.

4. *The definitions for the functions of an obtuse angle* have been made to grow organically out of the needs of the problem of solving triangles, in a way which seems both simple and natural, and which at the same time illustrates an important principle of mathematical procedure.

5. *The whole theory of triangles has been unified* by giving a central position to the area problem. As a consequence, almost all of the necessary equations present themselves spontaneously and in a connected fashion. The law of tangents is the only one which causes any difficulty in this respect. But the law of tangents also has been made to submit to a heuristic treatment, by

introducing the notion of the *form-ratio* of a triangle, and combining this notion with a direct geometric proof of the formulæ for $\sin A + \sin B$ and $\sin A - \sin B$.

6. *The numerical aspect of the work has been discussed very fully.* Directions for computation are given in great detail; most of the common sources of error are pointed out; and methods for detecting and correcting them are indicated. After a thorough discussion of the significance of the number of decimal places needed in a computation, the student is urged to train and use his judgment on this matter. He is given an opportunity to do this by supplying him with complete five- and three-place tables and a partial set of four-place tables.

7. *The slide rule is explained* with considerable detail and its use recommended. A number of other labor-saving devices are discussed.

8. *The examples have been selected with great care.* Examples without real significance have been avoided, and the numbers have been chosen so as not to lead to five-place calculations when such a show of accuracy would be absurd. Special efforts have been made to word the examples in such a way as to avoid ambiguity.

9. *The applications cover a wider field than usual,* and include problems in heights and distances, surveying, navigation, engineering, astronomy, and physics. But the examples involving such applications are not, as in most texts, introduced at random and without previous explanation. Every notion which is required for the solution of any example in the book is fully explained on the spot or in some earlier portion of the text.

10. *The use of a few new terms,* such as the *standard position* of an angle, odd and even *cardinal angles,* has helped to simplify materially the statement of a number of important results.

11. *The addition formulæ are presented in two different ways.* The first, more elementary method, is made to yield the general result by the help of mathematical induction. The second method, based on the notions of directed lines, line-segments, and angles, appears here in a very simple and elegant form.

12. *The articles on harmonic and wave motion* tend to show the student that Trigonometry has other applications besides the solution of triangles.

13. *A considerable amount of historical matter has been introduced,* not in the form of detached historical notes, but organically connected with the topic under discussion. Most of this matter was gathered from BRAUNMÜHL's *Vorlesungen über die Geschichte der Trigonometrie.* Professors CAJORI and KARPINSKI have kindly answered some questions of a historical nature about which we were in doubt.

14. *The type and the manner of spacing used in the tables* are the results of a number of experiments, the object being to produce a set of tables which should be as pleasant to the eye as possible. The tables are bound separately for various reasons. In order to make them easily legible, a certain size of page was necessary, and it was thought undesirable to use so large a page for the text itself. In the second place, it is a great advantage for the student if he can have his text and his tables open before him at the same time. In the third place, it is often desirable, in examinations, to allow students to use their tables without their books. Finally, a separation of the tables and text makes it easy to use this text with other tables, or these tables with other texts, thus providing a maximum of elasticity in organizing a course.

Many of the older texts on Trigonometry have been consulted during the preparation of this book, and the attempt has been made to learn from all of them. The works of SERRET, LÜBSEN, WIEGAND, CROCKETT, MORITZ, HALL and FRINK have been especially helpful. A few purely numerical examples have actually been taken from these and other texts without change, so as to reduce somewhat the task of computing the answers, and at the same time to make the answers more trustworthy. Most of the examples, however, are new; many of them are new in kind.

In conclusion, the author and editor wish to acknowledge their indebtedness to their colleagues at the University of Chicago for various helpful suggestions and criticisms.

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PLANE TRIGONOMETRY

AND

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PART ONE

SOLUTION OF TRIANGLES

CHAPTER I

THE OBJECT OF TRIGONOMETRY

1. Direct measurement of lines. One of the most common operations of practical geometry is that of measuring the distance between two points. In its simplest form this consists merely in the repeated application of some unit of measurement to the required distance.

The units of measurement most frequently used for this purpose are a foot rule, a yardstick, a surveyor's chain, tape lines of definite length, etc. Fractional parts of the unit are usually read from a graduated *scale*, engraved or stamped on the standard used. A familiar illustration of this is the scale of inches on an ordinary foot rule.

In spite of the apparent simplicity of this process, it is a matter of great practical difficulty to carry out such measurements with a high degree of precision. The sources of error are numerous and, in part, unavoidable. No instrument made by man is absolutely accurate. Thus, if we use a yardstick, it will not be absolutely straight, and it may be a trifle too long or too short. It will be very difficult to make sure that we are laying off the second yard of our distance exactly where the first yard ends. Consecutive positions of the yardstick will form angles with each other, which are not exactly equal to 180°. In fact, it is almost impossible to run a straight line of considerable length, with

any degree of accuracy, without the help of more complicated instruments, such as the transit described in Art. 2.

The graduated scales, used for measuring fractional parts of the unit of length, are also affected by various sources of inaccuracy, and it will be difficult for the observer to estimate accurately a fractional part of the smallest visible division on the scale.

Enough has been said to indicate just a few of the many difficulties encountered in the, apparently so simple, operation of measuring the length of a line, and to emphasize the fact that we must always regard the result of such a measurement as an *approximation*, even if the most refined instruments known to Science have been used.

The difference between rough and fine measurements is one of degree only. The more refined the method, the smaller will be the "probable error" and the closer the approach to the truth. But we can never be sure that a quantity has been measured with *absolute* precision.

EXERCISE I

1. What are some of the sources of inaccuracy in measuring the length of a table?

2. If you wish to measure the distance *diagonally* across a table, by means of a foot rule, what additional sources of inaccuracy will appear? Would a stretched cord be of some use in this connection?

3. How would you measure the distance diagonally across a room from one of the floor corners to the opposite corner of the ceiling? Do you know of any other method by which this distance might be obtained, except that of direct measurement?

4. How would you join two given points by a straight line (say for the purpose of constructing a fence), over a level piece of ground, if the distance is too great to enable you to stretch a cord? Do you know of any property of the line of sight which might help in the solution of such a problem?

5. If you attempt to measure two different distances, of which one is about ten times as great as the other, using the same foot rule and the same method of measurement in both cases, which of these two distances will probably be obtained with greater accuracy? Why?

6. What difficulties arise, and how would you attempt to meet them, if you were asked to measure the horizontal distance between two points on an uneven piece of ground?

7. Suppose you have measured a distance (say the edge of a table) with great care and have found it to be equal to 4 feet and $9\frac{3}{8}$ inches. Is this the *exact* length of the table, or may it be a small fraction of an inch greater or less?

8. If you were to repeat the measurement with still greater care, making use of a more perfect standard of length, is it likely that you would find exactly the same result as before?

9. In any such measurement can you ever attain *absolute* accuracy? If not, why not?

10. Is there any way of *knowing* whether a measurement is *absolutely* accurate? Is there any way of knowing whether it is accurate to within a certain limit of accuracy, say one tenth of an inch?

11. If a distance has been measured by a process which may be relied upon to give a result accurate only to one one-hundredth of an inch, is it desirable, proper or honest, to give the result expressed in decimal parts of an inch to more than two decimal places? Why not? In performing calculations based upon such measurements, how many decimal places should we ordinarily carry?

12. In measuring distances by means of metal rods, when great accuracy is required, changes of temperature must be taken into account. Why?

13. With what units of length are you familiar?

14. Gather from an encyclopedia what you can concerning the "standard yard" kept at Washington.

15. What is the metric system? What are the relations to each other of the units called millimeter, centimeter, decimeter, and meter? What is the length of a meter in inches?

16. Can you see any reason why the metric system should be preferable to the English system of weights and measures?

2. Direct measurement of angles. The operation of measuring angles is scarcely less important than that of measuring distances. A **protractor** is an instrument used for this purpose. In its simplest form, a protractor consists of an arc of a circle graduated to degrees (Fig. 1). A

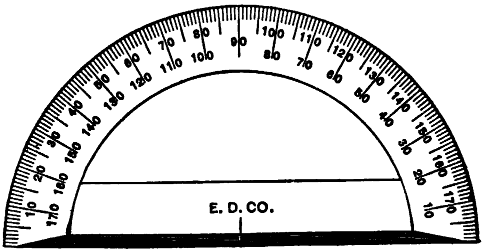


Fig. 1

mere inspection of the instrument will enable the student to see how angles may be measured and constructed by means of it.

The unit usually employed in measuring angles is the ninetieth part of a right angle, and is called a *degree.* A degree is divided into sixty equal parts, each of which is called a *minute*, and each minute is subdivided into sixty *seconds.* Thus

$$60 \text{ seconds } (60'') = \text{one minute } (1'),$$
$$60 \text{ minutes } (60') = \text{one degree } (1^\circ),$$
$$90 \text{ degrees } (90^\circ) = \text{one right angle.}$$

Very frequently, the angles smaller than one degree are expressed as decimal parts of a degree instead of in minutes and seconds.

FIG. 2

For the purpose of measuring angles in the field, surveyors make use of an instrument called a **transit** or **theodolite.** The essential parts of this instrument are (cf. Fig. 2):

1. a horizontal graduated circle;

2. a movable circular plate adjusted so as to be capable of rotation around the center of the horizontal graduated circle;

3. an index attached to the movable plate in such a way as to enable the ob-

server to read off the amount of its rotation with reference to the fixed horizontal circle;

4. two standards attached to the movable plate and carrying a horizontal axis to which is attached a telescope and also, in a complete instrument, a vertical graduated circle, used for measuring angles whose sides lie in a vertical plane.

The transit is usually supported on a tripod. If we wish to measure the angle between two horizontal lines, the tripod is placed over the vertex of the angle and the telescope is pointed toward some point on one of the sides of the angle. The index will then point to a definite division on the horizontal circle. The operation of ascertaining the division of the circle toward which the index is pointing, is known as "reading the circle." After reading the circle and noting the result, the telescope is directed toward a point on the other side of the angle. The difference between the two readings of the horizontal circle will give the magnitude of the angle.

In a similar way the vertical circle, which is attached to the axis of the telescope, makes possible the measurement of angles whose sides lie in a vertical plane.

Both circles are usually graduated to whole degrees. The index, in most instruments, is not a simple pointer, but a so-called **vernier**, an ingenious device which enables the observer to determine the reading of the circle to within a small fraction of a degree, even if the circle is graduated only to whole degrees. In the most accurate instruments, the vernier is replaced by a **reading microscope**.

It is obviously very important that the horizontal circle be exactly horizontal, and that its center be exactly over the vertex of the angle which is to be measured. To help in making these adjustments, the surveyor uses a **spirit level** and a **plumb line**.

Most transits are also supplied with a **compass**, which enables the observer to determine the absolute directions of the lines which he is surveying.

EXERCISE II

1. Use a protractor to draw angles of 10°, 20°, 31°, 47° 30′, 67°, 78°, 86°.

2. Draw five angles at random and measure them as accurately as possible with your protractor.

3. Draw a triangle at random, measure its angles and find their sum. What should this sum be? If you have obtained a different result for the sum, what are the reasons?

4. Construct, out of cardboard, an instrument embodying the principle of the transit, substituting for the telescope some other method of taking a sight.

5. Why is it important that the horizontal circle of a transit should be truly horizontal? How does the spirit level enable us to make it so? Study the article on the spirit level in some encyclopedia or in a treatise on surveying.

6. Study the articles on *vernier*, *micrometer*, *reading microscope*, *compass*, in an encyclopedia or in some appropriate treatise, and write an abstract of the same.

7. Describe the sources of inaccuracy which you think may arise in the measurement of an angle by means of a protractor or theodolite.

8. Deduce rules for converting minutes and seconds into decimal parts of a degree and *vice versa*.

9. Apply this rule to the angles

$$30° 20', \ 10° 45', \ 8° 40' 20'', \ 3° 8' 2''.$$

10. Express the following angles in degrees, minutes, and seconds:

$$23°.14, \ 18°.25, \ 46°.235.$$

3. The impossibility of finding all distances by direct measurement. We have discussed briefly the direct methods for measuring distances and angles. It is clear from our discussion that it is very *difficult* to measure great distances in that way. But there are many cases in which it is altogether *impossible* to apply such direct methods. How, for example, should we proceed to find the distance through a mountain or across an extensive valley? How shall we find the distance from New York to London, or from the earth to the moon, by direct measurement?

It is clear that, if we wish to answer such questions at all, we shall have to devise some method different from that of direct measurement.

The attempt to solve, by indirect methods, problems whose direct solution is inconvenient or impossible usually leads to great advances in Science. The most important theorems of elementary geometry were probably first discovered by the Ancients in their attempts to devise convenient and practical methods for the measurements which they found necessary for the purposes of their everyday life. It is generally believed, for instance, that the Egyptians became expert geometricians and surveyors at an early period of their history, because it was so important

for them to be able to reëstablish the boundary lines of their lands after the effacement of their marks by the annual inundations of the Nile.

The earliest Greek philosophers and mathematicians were pupils of the Egyptians, and some of their first achievements were connected with problems of the particular kind which we are now discussing. Thus it is reported that THALES OF MILETUS (about 600 B.C.) measured the height of a pyramid by measuring the length of its shadow, at the instant when the shadow of a vertical stick by its side was exactly as long as the stick itself. The height of the pyramid would then be equal to the length of its shadow at that moment.

This method has the inconvenient feature of compelling the observer to wait (many hours perhaps) for the right moment. According to a report by PLUTARCH, THALES also devised a second method which avoided this inconvenience. This method involves the use of the simplest properties of similar triangles and may be illustrated by means of the following example:

We place a stick 5 feet high into the ground near a building whose height we wish to find. At any convenient moment we measure the

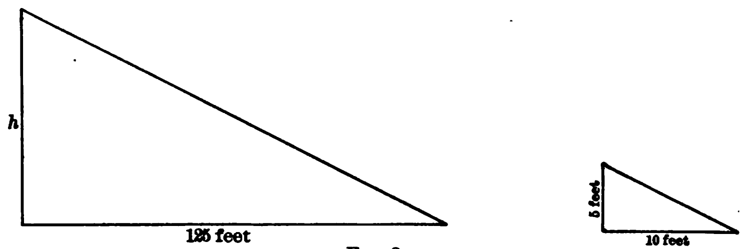


FIG. 3

length of the shadows of the stick and of the building. Suppose we find in this way that the shadow of the building is 125 feet long at the moment when the shadow of the stick is 10 feet long. If h denotes the height of the building, we shall have (Fig. 3)

$$h : 125 = 5 : 10,$$

whence $$h = 62.5 \text{ feet}.$$

According to EUDEMUS (about 300 B.C.), one of the earliest writers on the history of mathematics, THALES also invented a method for measuring the distance from the shore to a ship at sea. Although Eudemus does not describe Thales's method in detail, he says enough to lead us to conclude that it was essentially as follows: Let BL be a tower, say a lighthouse on the shore, and

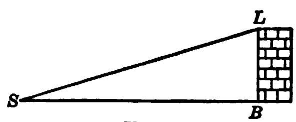


FIG. 4

let S be the ship at sea. Measure the angle BLS and the height of the tower. Construct a similar triangle $B'L'S'$ on the drawing board and measure $B'S'$ and $B'L'$. Then BS can be found from the proportion

$$BS : BL = B'S' : B'L'.$$

4. The graphic method. The examples discussed in Art. 3 show how we may solve many problems of practical geometry by indirect measurements. We did not measure directly the quantity which we were seeking, but some other related quantities, and ultimately, by means of these relations, we determined the desired quantity itself.

But these same examples may serve to illustrate another point. The solutions which we gave are examples of the **graphic method**, that is, of a process which makes use of drawing instruments, of accurate geometric constructions and measurements, for the purpose of obtaining the values of the unknown quantities. Such graphical methods are often extremely valuable and have been developed in recent times, in connection with other parts of mathematics, so as to give rise to very important results.

The graphic solution of any problem about triangles will finally reduce to an application of certain theorems of geometry which state that a triangle is determined and may be constructed when certain ones of its six parts (sides and angles) are given. It is clear, then, that these theorems are particularly important for the graphic method. Some of the following questions have been chosen for the purpose of aiding the student to refresh his memory in regard to these matters.

EXERCISE III

In the following examples and throughout the book we shall usually denote the angles of a triangle by A, B, C and the sides opposite to these angles by a, b, c, respectively. The student should be provided with a protractor, a pair of compasses, and a ruler divided decimally, say into centimeters and millimeters. All constructions and measurements should be made as carefully as possible.

1. Given $a = 3.72$, $b = 4.91$, $c = 2.56$. Find the angles.

2. Given $a = 4.27$, $B = 35°$, $C = 69°$. Find the remaining parts of the triangle.

3. Given $b = 5.63$, $c = 6.71$, $A = 27°$. Find the remaining parts of the triangle.

4. Given $a = 4.23$, $b = 5.16$, $A = 55°$. Find the remaining parts of the triangle.

5. When a, b, c are given at random, can we *always* find a triangle of which a, b, c are the sides, or is there some restriction on the possible values of a, b, c?

6. If a, b, A are given, there may be one or two solutions, or no solution. Discuss these cases.

7. Is a triangle determined when we know the magnitudes of its three angles? Why? When the three angles of a triangle are given, have we obtained essentially more information than if only two of them are given? Why? Is the third angle of a triangle independent of the other two?

8. What do you mean by similar triangles?

9. Under what conditions are two triangles similar?

10. Use the shadow method of Thales (Art. 3) to measure the height of some building in your neighborhood.

11. By means of a transit, or else by means of the home-made instrument of cardboard suggested in Ex. 4 of Exercise II, find the distance of some object situated on the opposite side of the street from your home, without crossing the street. Afterward check your result by direct measurement.

12. How may a person on board ship find his distance from a building on the shore if he knows its height?

5. The desirability of an arithmetical method for solving triangles. We have seen that it is a rather simple matter to solve a triangle by the graphic method. But we can hardly feel altogether satisfied with the graphic solution, for it will clearly not permit us to reach any great degree of accuracy. To be sure, by making our drawings on a very large scale, we might lay off distances with considerable precision, but we should still encounter the difficulty of accurately plotting angles. Clearly it would require extraordinary skill and exceedingly fine instruments to enable us to draw an angle so accurately that its error should not exceed one minute of arc. Many other circumstances combine to make a graphical solution unsatisfactory if a high degree of precision is required.

The main value of the graphic method lies in furnishing a solution whose approximate correctness is apt to be apparent to

the eye, and which may therefore, in almost all cases, serve as a check on the more complete solution obtained in some other way.

But the lack of accuracy is only one of the defects of the graphic method, although perhaps the most important one from a practical point of view. The other defect which we wish to emphasize is more of a theoretical nature. The parts of a triangle (its sides and angles) are usually given as *numbers* (so many feet, so many degrees). It seems natural, therefore, to suppose that there must exist some *arithmetical process* for the solution of triangles.

Trigonometry enables us to find the unknown parts of a triangle by arithmetical processes.

This statement must *not* be regarded as a complete definition of trigonometry. We shall see later that the solution of triangles by arithmetical methods constitutes only a part, although an important part, of trigonometry.

EXERCISE IV

1. What are the principal sources of inaccuracy in solving a triangle by the graphic method?

2. We wish to construct an isosceles triangle, given the angle at the vertex and the altitude. Suppose that the error made in constructing the angle is $5'$. Will the effect of this error on the base of the triangle, as obtained from the construction, be different for different altitudes? Will it be greater or less for the higher triangles?

3. If you wish to find a point as the intersection of two straight lines, are you likely to obtain a more accurate result if the two lines are at right angles, or if they are nearly parallel?

4. A side and two adjacent angles of a triangle are given, and its other parts are to be found graphically. Is such a graphic solution likely to be accurate if both of the angles are very small? If the sum of the given angles is very close to $180°$?

5. Formulate, in your own words, the distinction between an arithmetical and a graphical process.

CHAPTER II

THE TRIGONOMETRIC FUNCTIONS OF ACUTE ANGLES

6. The necessity of introducing new ideas. At the close of Chapter I we formulated the problem of devising arithmetical methods for the solution of triangles. But, if we think of the many shapes which a triangle may assume, this problem appears to be a most formidable one. We shall, therefore, for the present, confine our attention to the comparatively simple case of a *right triangle.* We have good reason to suppose that, if we succeed in solving our problem for all right triangles, we shall be able to deal later with the general case also, since every triangle may be decomposed into two right triangles.

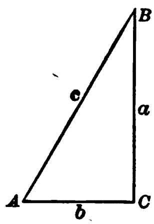


FIG. 5

Let us then consider a triangle ABC, right-angled at C (Fig. 5), and let us denote the sides opposite the angles A, B, C by a, b, c, respectively. We are already acquainted with two relations which will assist us in the solution of our problem, namely:

$$(1) \qquad A + B = 90°$$

and the theorem of PYTHAGORAS,

$$(2) \qquad a^2 + b^2 = c^2.$$

The first of these equations enables us to calculate one of the acute angles if the other one is given. The second provides a method for calculating any one of the three sides if the other two are given.

But we have nothing as yet which will enable us to find the angle A if two of the sides (say a and b) are given, although the triangle is clearly determined by these sides and might be constructed by geometry. The equations (1)

and (2), unaided by other relations, are *obviously* inadequate for this purpose, since (1) is a relation between the *angles* A and B alone, while (2) involves only the *sides* of the triangle. Now, clearly, a statement which is only concerned with the *angles* of a triangle cannot convey any positive information about the *sides*, and *vice versa.*

In order that we may be able to solve a right triangle by arithmetical processes, there must, therefore, be added to equations (1) *and* (2) *certain other relations, in which the sides and angles shall not be separated, but in which they shall occur simultaneously.*

The discovery of such relations and their adequate formulation is the foundation upon which all of trigonometry must finally rest.

A simple illustration will make clear the nature of these new relations. The numerical measure for the steepness of an inclined plane, say a mountain road, may be given in two ways. We may say that the road makes a certain angle A (say 5°) with the horizontal plane, or we may say that it rises a certain number of feet (say 87.5 feet) in a horizontal distance of 1000 feet.

The quotient, $\frac{87.5}{1000}$, is technically known as the *slope, grade,* or *gradient* of the road. The gradient is clearly connected with the angle A in such a way that, if A should vary, to every value of the angle there corresponds a definite value of the gradient and *vice versa.* Thus, if AB (Fig. 6) represents the road, and if a and b are both expressed in feet, the gradient of AB is equal to $\frac{a}{b}$. The value of this quotient changes with the angle A, so that for different angles we find different values for the gradient.

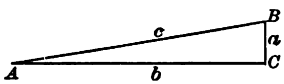


FIG. 6

The general question before us may be formulated as follows. If the acute angle A of the right triangle ABC becomes larger or smaller, what effect will such a change have upon the sides? And, conversely, if the sides of a triangle change, what effect will this have on the angles?

Now, since similar triangles have their corresponding angles equal, it is clear that there are *some* changes in the

lengths of the sides which produce *no* change in the angles. In fact, if all of the sides of a triangle are magnified in the same ratio, the angles are not changed at all.

In the two right triangles ABC and $A'B'C'$ (Fig. 7) we have

$$\frac{a}{c}=\frac{a'}{c'},\ \frac{b}{c}=\frac{b'}{c'},\ \frac{a}{b}=\frac{a'}{b'},\ \text{etc.},$$

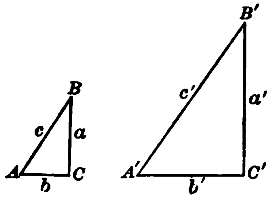


FIG. 7

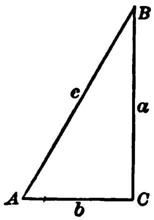
if the angle at A' is equal to that at A. But if the angle A' is different from A, the ratios $\frac{a'}{c'}$, $\frac{b'}{c'}$, $\frac{a'}{b'}$, etc., will not be equal to $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, etc., respectively. For if they were, corresponding pairs of sides of the two triangles would have the same ratio, the triangles would be similar, and, contrary to our hypothesis, angle A' would have to be equal to angle A.

Consequently, while the *lengths* of the individual sides of a right triangle have nothing to do with the size of its angles, the *ratios* of these lengths *are* connected with the magnitude of the angles in a very intimate fashion. In fact, so close is this relation that the values of the ratios $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, etc., may be determined (by construction) as soon as the acute angle A has been chosen, and conversely; if one of these ratios is given, we can find (by construction) one, and only one, corresponding acute angle A.

7. Definitions of the trigonometric functions of an acute angle. We have seen that the values of the ratios $\frac{a}{c}$, $\frac{b}{c}$, etc., of the sides of a right triangle are closely bound up with the magnitude of the angle A. If the angle changes, each of these ratios changes and *vice versa*.

Now, *one variable quantity is called a* ***function*** *of another, if they are so related that any change in the latter produces a corresponding change in the former.*

Consequently, each of the six ratios $\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{a}, \frac{c}{b}$, determined by the sides of the right triangle ABC, is a *function* of the angle A, because any change in A produces a corresponding change in each of those ratios. Each of these functions has received a name and a symbol. The reason for choosing these names will appear later (see Arts. 10 and 70), and cannot be discussed with profit at the present moment.



We proceed to give the formal definitions of the six trigonometric functions of an acute angle A.

Construct any right triangle (cf. the Fig.), *one of whose acute angles is equal to the given angle A.* Of the two legs of this right triangle, that one which passes through the vertex of the angle A is said to be* ***adjacent*** *to A. The other leg is said to be* ***opposite*** *to A, and the third side of the triangle is called its* ***hypotenuse***.

The **sine of A** is the ratio of the **opposite side** to the **hypotenuse.**

The **cosine of A** is the ratio of the **adjacent side** to the **hypotenuse.**

The **tangent of A** is the ratio of the **opposite side** to the **adjacent side.**

The **cotangent of A** is the ratio of the **adjacent side** to the **opposite side.**

The **secant of A** is the ratio of the **hypotenuse** to the **adjacent side.**

The **cosecant of A** is the ratio of the **hypotenuse** to the **opposite side.**

---

* We have seen in Art. 6 that the size of this right triangle is absolutely of no consequence, since any two triangles of this kind are similar, so that the corresponding ratios for the two triangles will be equal.

In symbols we may write these definitions as follows:

$$(1)\qquad \begin{aligned} \sin A &= \frac{a}{c}, & \cos A &= \frac{b}{c}, & \tan A &= \frac{a}{b}, \\ \csc A &= \frac{c}{a}, & \sec A &= \frac{c}{b}, & \cot A &= \frac{b}{a}. \end{aligned}$$

These symbols are written abbreviations of the names of the functions. In speaking, the symbols are pronounced as though the name of the function had been written out in full. Thus, tan A is pronounced *tangent of* A or *tangent* A; csc A is pronounced *cosecant of* A or *cosecant* A, etc.

In defining the trigonometric functions, we made use of the numerical measures of certain line-segments, namely, of the sides of a right triangle. Now, the numerical measure of a line-segment changes if the unit of measurement is changed. One might, therefore, expect the values of the trigonometric functions of an acute angle to change with every change of the unit of length. But this is *not* the case. Owing to the fact that only *ratios* of these line-segments appear in equations (1), the functions sin A, cos A, etc., are found to have the same value whether the line-segments a, b, c are measured in feet, inches, or in terms of any other unit of length.

Consider, for instance, a right triangle for which $a = 3$ feet, $b = 4$ feet, $c = 5$ feet. According to (1) we have $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, etc. Let us now introduce the inch as unit of length instead of the foot. The numerical measures of the sides of the triangle will now be $a = 36$, $b = 48$, $c = 60$. According to (1) we shall *now* find $\sin A = \frac{36}{60}$, $\cos A = \frac{48}{60}$, etc. But $\frac{36}{60} = \frac{3}{5}$, $\frac{48}{60} = \frac{4}{5}$, etc., so that we obtain precisely the same values for sin A, cos A, etc., whether the sides of the triangle be expressed in feet or inches.

If the trigonometric functions were *concrete* numbers, that is, if they were the numerical measures of some kind of concrete quantity such as a length, an area, or a volume, their values would change every time that a change is made from one unit of length to another. We have just seen that this is not the case. Therefore, *the trigonometric functions are pure or abstract numbers.**

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* This same fact is sometimes (somewhat inadequately) expressed by the statement that *the trigonometric functions are ratios.*

8. The practical need of tables giving the values of the trigonometric functions. The trigonometric functions just defined will enable us to find the unknown parts of a right triangle when certain parts are given, provided only that we can devise a practical method for actually obtaining the numerical values of these functions for any given acute angle. Now, the values of the functions have been calculated by mathematicians and the results have been collected in tabular form for convenient use. From the practical point of view, therefore, it only remains to explain the arrangement and the use of the tables.

To the more scientifically inclined student, however, the question will immediately suggest itself as to how these useful tables were actually computed. We shall reserve the answer to this question for the second part of the book. The following examples, however, will show how a table of trigonometric functions may be prepared by the graphic method, provided that no very high degree of accuracy be required.

EXERCISE V

1. Find the functions of 40° by the graphic method.

Solution. With the help of a protractor construct the angle PAQ (Fig. 8) equal to 40°. Choose any point B on AQ and drop a perpendicular BC from B to AP. Measure the distances BC, AC, and AB. Then will

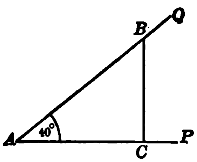

Fig. 8

$$\text{(1)} \qquad \sin 40^\circ = \frac{CB}{AB}, \quad \cos 40^\circ = \frac{AC}{AB}.$$

Although the point B might be chosen *anywhere* on AQ, it will be especially convenient to make AB equal either to one unit, ten units, or one hundred units. For, as equations (1) show, we have to divide by AB, and if AB is equal to 1, 10, or 100, we avoid the long division which would otherwise be necessary.

Let us, therefore, make $AB = 10$ centimeters. We should then find by measurement,

$$CB = 6.4 \text{ cm.}, \; AC = 7.7 \text{ cm.}$$

According to (1), therefore,

$$\sin 40^\circ = \frac{6.4}{10} = 0.64, \quad \cos 40^\circ = \frac{7.7}{10} = 0.77.$$

Further we find

$$\tan 40° = \frac{6.4}{7.7} = 0.83, \quad \cot 40° = \frac{7.7}{6.4} = 1.20,$$

$$\sec 40° = \frac{10}{7.7} = 1.30, \quad \csc 40° = \frac{10}{6.4} = 1.56.$$

We should obtain a more accurate result for tan 40°, and more conveniently, if we were to use another triangle, making this time $AC = 10$ cm. Measurement would then give $BC = 8.4$ cm., and

$$\tan 40° = \frac{8.4}{10} = 0.84.$$

2. Find the functions of the following angles by the graphic method:

(*a*) 10°. (*b*) 15°. (*c*) 20°. (*d*) 70°.

Construct carefully each of the following right triangles, measure the angles, and find the six functions of each acute angle.

3. $a = 3, \; b = 4, \; c = 5.$

4. $a = 5, \; b = 12, \; c = 13.$

5. $a = 8, \; b = 15, \; c = 17.$

6. Construct and measure an acute angle whose sine is equal to $\frac{1}{3}$.

7. Construct and measure an acute angle whose tangent is equal to $\frac{2}{3}$.

8. Construct and measure an acute angle whose cosine is equal to $\frac{4}{5}$.

9. Can you think of two right triangles (Fig. 7) with different angles A and A', for which the sides a and a' are nevertheless equal?

10. Can you conceive of two right triangles (Fig. 7) with different angles A and A', for which the ratios $\frac{a}{b}$ and $\frac{a'}{b'}$ are nevertheless equal?

11. Why, then, do we speak of the ratio $\frac{a}{b}$ as a function of A? Why do we not introduce a or ab as a function of A?

12. Assuming that we have access to a table of the values of the trigonometric functions, show how we might solve each of the following problems. To find the remaining parts of a right triangle when the following parts are given.

I. $a, \; b, \; C = 90°.$ III. $a, \; c, \; C = 90°.$
II. $a, \; A, \; C = 90°.$ IV. $c, \; A, \; C = 90°.$

13. Show that neither the sine nor the cosine of an acute angle can ever be greater than unity.

14. Show that the tangent of an acute angle may have any positive value whatever. Similarly for the cotangent.

15. What restrictions, if any, are there on the values which the secant and cosecant of an acute angle may assume?

9. Relations between the six trigonometric functions of an acute angle. The preceding discussion suffices to indicate the importance of constructing a table of the values of the trigonometric functions. The task of computing these tables may be abbreviated very considerably by noting that the six functions are not independent of each other. In fact, we have (Fig. 9)

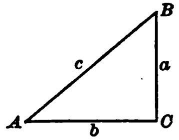

FIG. 9

$$(1)\qquad \sin A = \frac{a}{c},\quad \csc A = \frac{c}{a},$$
$$\cos A = \frac{b}{c},\quad \sec A = \frac{c}{b},$$
$$\tan A = \frac{a}{b},\quad \cot A = \frac{b}{a},$$

so that we obtain at once the relations

$$\csc A = \frac{1}{\sin A},\quad \sec A = \frac{1}{\cos A},\quad \cot A = \frac{1}{\tan A},$$
$$\sin A = \frac{1}{\csc A},\quad \cos A = \frac{1}{\sec A},\quad \tan A = \frac{1}{\cot A},$$

or,

$$(2)\qquad \mathbf{\sin A \csc A = 1,\quad \cos A \sec A = 1,\quad \tan A \cot A = 1.}^*$$

But, two numbers whose product is equal to unity are called *reciprocals* of each other. Therefore, equations (2) are equivalent to the following statement:

The sine and cosecant, the cosine and secant, and finally the tangent and cotangent, of an acute angle are reciprocals.

Clearly, knowledge of this fact reduces greatly the labor of computing tables of the functions. For, having found the values of the three functions, sine, cosine, and tangent, the values of the remaining three can be obtained from these by computing their reciprocals.

But there are other relations besides (2) which enable us to reduce still further the labor involved in constructing a table of the trigonometric functions. We have

$$\tan A = \frac{a}{b}.$$

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* Note that $\sin A \csc A$ is written for $\sin A \times \csc A$ just as ab is written for $a \times b$.

If we divide both numerator and denominator of the fraction $\frac{a}{b}$ by c, we find

$$\tan A = \frac{a/c}{b/c}.$$

But we have, by definition (cf. equations (1))

$$\frac{a}{c} = \sin A, \quad \frac{b}{c} = \cos A,$$

and therefore

$$\tan A = \frac{\sin A}{\cos A}. \tag{3}$$

Of course, since $\cot A$ is the reciprocal of $\tan A$, we also have

$$\cot A = \frac{\cos A}{\sin A}. \tag{4}$$

The relations (2), (3), (4) enable us to calculate the values of all six functions when the sine and cosine are known. But it actually suffices to know the sine. For, if we divide both members of the familiar equation

$$a^2 + b^2 = c^2 \tag{5}$$

by c^2, we find

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

But, by definition, we have

$$\frac{a}{c} = \sin A, \quad \frac{b}{c} = \cos A,$$

so that (5) becomes

$$\sin^2 A + \cos^2 A = 1, \tag{6}$$

where $\sin^2 A$ and $\cos^2 A$ have been written, as is customary, for $(\sin A)^2$ and $(\cos A)^2$, respectively.

It follows from relations (2), (3), (4), and (6) that, if we know the value of a single one of the six trigonometric functions of an acute angle, the values of the remaining five may be computed. The detailed proof of this statement will be left to the student in some of the examples given below.

EXERCISE VI

In each of the following twelve examples, the value of one function of the acute angle A is given. Find the values of the remaining functions.

1. $\sin A = \frac{3}{5}$.	**4.** $\cot A = 2$.	**7.** $\sin A = x$.	**10.** $\cot A = x$.
2. $\cos A = \frac{5}{13}$.	**5.** $\sec A = \frac{13}{12}$.	**8.** $\cos A = x$.	**11.** $\sec A = x$.
3. $\tan A = 1$.	**6.** $\csc A = \frac{17}{8}$.	**9.** $\tan A = x$.	**12.** $\csc A = x$.

13. Construct an isosceles right triangle and make use of this figure for the purpose of computing the functions of 45°.

14. Divide an equilateral triangle into two right triangles by dropping a perpendicular from one of its vertices to the opposite side. Make use of this figure for the purpose of computing the functions of 30° and 60°.

15. Collect in a table the results of Exs. 13 and 14.

16. Prove the formula $\sec^2 A = 1 + \tan^2 A$.

17. Prove the formula $\csc^2 A = 1 + \cot^2 A$.

18. Prove that the sine, tangent, and secant of an angle increase when the angle grows from 0° to 90°.

19. Prove that the cosine, cotangent, and cosecant of an angle decrease when the angle grows from 0° to 90°.

20. Prove that $\tan A < 1$, $\cot A > 1$ if $A < 45°$, and that $\tan A > 1$, $\cot A < 1$ if $A > 45°$.

21. Show that each of the six functions may be expressed either as a product or as a quotient of two of the others.

10. Relations between the functions of complementary angles. As a result of the relations discussed in the preceding article, the problem of computing the values of the six trigonometric functions for every angle between 0° and 90° has been reduced to that of computing these values for a single one of these functions. But we may reduce the problem still further by observing the relations between the functions of the two acute angles of the same right triangle.

We have (cf. Fig. 10)

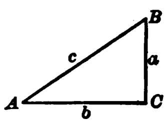

Fig. 10

$$\sin A = \frac{a}{c}, \quad \cos B = \frac{a}{c},$$

$$\cos A = \frac{b}{c}, \quad \sin B = \frac{b}{c},$$

$$\tan A = \frac{a}{b}, \quad \cot B = \frac{a}{b},$$

$$\cot A = \frac{b}{a}, \quad \tan B = \frac{b}{a},$$

$$\sec A = \frac{c}{b}, \quad \csc B = \frac{c}{b},$$

$$\csc A = \frac{c}{a}, \quad \sec B = \frac{c}{a},$$

and therefore

$$(1) \quad \begin{array}{lll} \sin A = \cos B, & \tan A = \cot B, & \sec A = \csc B, \\ \cos A = \sin B, & \cot A = \tan B, & \csc A = \sec B. \end{array}$$

Since the angles A and B are complementary, we may write these equations as follows:

$$(2) \quad \begin{array}{ll} \sin(90° - A) = \cos A, & \cot(90° - A) = \tan A, \\ \cos(90° - A) = \sin A, & \sec(90° - A) = \csc A, \\ \tan(90° - A) = \cot A, & \csc(90° - A) = \sec A. \end{array}$$

An easy way to remember these formulæ is as follows: Let the six functions be grouped into three pairs: sine and cosine, tangent and cotangent, secant and cosecant. Let us speak of either function of one of these pairs as the *cofunction* of the other. Then, the six formulæ (2) are all included in the following statement.

Any trigonometric function of the complement of an angle A is equal to the cofunction of A.

It is apparent that this theorem will enable us to find the trigonometric functions of any acute angle greater than 45°, if we know the functions of all angles less than 45°. Thus, for instance, tan 75° is equal to cot 15°, sin 82° is equal to cos 8°, etc. As a consequence of this fact it is possible to reduce the space occupied by the tables of the functions to exactly half of what would otherwise be necessary.

The relation between the functions of complementary angles is also important in another respect. It is this relation which has given rise to the words cosine, cotangent, and cosecant. The cosine is the sine of the complement. At a time when Latin was still the universal language of the scientific world, the cosine was therefore called *complementi sinus.*

This was later (in the seventeenth century) contracted to *cosinus*. The words cotangent and cosecant originated in the same manner.

EXERCISE VII

1. Express as functions of the complementary angles

$$\sin 37°, \cos 62°, \tan 13°, \cot 75, \sec 12° 15', \csc 55° 37'.$$

2. If the table of values of the functions is so arranged as to give only the functions of angles less than 45°, how may we obtain the values of

$$\sin 57°, \cos 63° 15', \tan 75° 12', \cot 67° 18'?$$

3. What acute angle is that whose sine is equal to the sine of its complement?

4. Find an acute angle for which $\tan A = \cot(45° + A)$.

HINT. Substitute for $\tan A$ its equal $\cot(90° - A)$ and note that two acute angles with the same cotangent are equal to each other.

5. Find an acute angle for which $\sin 2A = \cos(45° - A)$.

6. Find an acute angle for which $\cot 3A = \tan 2A$.

7. Find an acute angle for which $\cos A = \sin 6A$.

8. Find an acute angle for which $\sec 2A = \csc 7A$.

11. The values of the functions of 0°, 30°, 45°, 60°, 90°. While we have decided to postpone the general question of the arithmetical calculation of the trigonometric functions, we have already performed this calculation for a few special angles, viz.: 30°, 45°, 60° (cf. Exs. 13, 14 of Exercise VI). The figures there suggested, and the results are as follows:

$$(1)\quad \begin{cases} \sin 45° = \cos 45° = \dfrac{a}{a\sqrt{2}} = \dfrac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}, \\ \tan 45° = \cot 45° = 1, \\ \sec 45° = \csc 45° = \sqrt{2}. \end{cases}$$

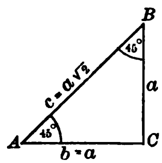

FIG. 11

$$(2)\quad \begin{cases} \sin 30° = \cos 60° = \dfrac{\frac{1}{2}c}{c} = \dfrac{1}{2}, \\ \cos 30° = \sin 60° = \frac{1}{2}\sqrt{3}, \\ \tan 30° = \cot 60° = \dfrac{\frac{1}{2}c}{\frac{1}{2}c\sqrt{3}} = \dfrac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}, \\ \cot 30° = \tan 60° = \sqrt{3}, \\ \sec 30° = \csc 60° = \dfrac{c}{\frac{1}{2}c\sqrt{3}} = \dfrac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}, \\ \csc 30° = \sec 60° = 2. \end{cases}$$

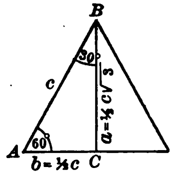

FIG. 12

An acute angle of a right triangle can never be 0° or 90°, so that the definitions of Art. 7 are not applicable to such angles. The acute angle A may, however, *approach* either 0° or 90° *as a limit*, and if it does, its functions in some cases approach definite finite limits. By sin 0°, cos 0°, sin 90°, cos 90°, etc., we mean such limits whenever they exist.

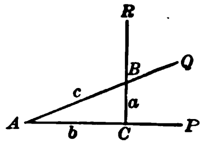

FIG. 13

In Fig. 13, let the angle $A = PAQ$ be thought of as decreasing toward the limit zero as a result of the rotation of the side AQ around A as a center, while the side AP remains fixed. Through any point C of AP draw CR perpendicular to AP, and denote by B the intersection of CR with the rotating line AQ.

As the angle A approaches zero, the side a approaches zero and c approaches b. Consequently

$$\sin A = \frac{a}{c} \text{ approaches } \frac{0}{b} \text{ or } 0, \quad \cos A = \frac{b}{c} \text{ approaches } \frac{b}{b} \text{ or } 1,$$

$$\tan A = \frac{a}{b} \text{ approaches } \frac{0}{b} \text{ or } 0, \quad \sec A = \frac{c}{b} \text{ approaches } \frac{b}{b} \text{ or } 1.$$

In this sense we may say that

$$\sin 0° = 0, \ \cos 0° = 1, \ \tan 0° = 0, \ \sec 0° = 1.$$

The function $\cot A = \frac{b}{a}$ *has no limit when A approaches zero.* For, the ratio $\frac{b}{a}$ grows larger and larger as A approaches zero, since b remains fixed while a grows smaller and smaller. Clearly, this ratio may be made larger than any number however great, by choosing A and, hence, a small enough. This is expressed in symbolic language as follows:

$$\cot 0° = \infty,$$

or in words: *The cotangent of an acute angle increases without bound when the angle approaches zero as a limit.*

A similar argument holds for $\csc A = \frac{c}{a}$, since c approaches

b and a approaches zero. Hence, in symbolic language, we have

$$(3)\qquad \begin{cases} \sin 0^\circ = 0, & \tan 0^\circ = 0, & \sec 0^\circ = 1, \\ \cos 0^\circ = 1, & \cot 0^\circ = \infty, & \csc 0^\circ = \infty. \end{cases}$$

By a similar argument the student may deduce the following results:

$$(4)\qquad \begin{cases} \sin 90^\circ = 1, & \tan 90^\circ = \infty, & \sec 90^\circ = \infty, \\ \cos 90^\circ = 0, & \cot 90^\circ = 0, & \csc 90^\circ = 1, \end{cases}$$

the exact meaning of each of which should be expressed in words as in the cases which have just been treated *in extenso*.

CHAPTER III

SOLUTION OF RIGHT TRIANGLES BY NATURAL FUNCTIONS

12. Arrangement and use of the table of natural functions. The numerical values of the trigonometric functions are usually called the **natural functions** to distinguish them from the *logarithms* of these functions which we shall study later.

Table * V gives the numerical values of the sine, cosine, tangent, and cotangent to four decimal places for every tenth of a degree from 0° to 90°. The values of the secant and cosecant are omitted because they are not used very frequently. They may of course be calculated, whenever necessary, by the formulæ

$$\sec A = \frac{1}{\cos A}, \quad \csc A = \frac{1}{\sin A}. \qquad \text{(Art. 9)}$$

The following is a sample portion of the table. Only this part of the table will be required for the following illustrative examples.

ANGLE	N SIN	d	N TAN	d	N COT	d	N COS	d	
35°.0	0.5736	14	0.7002	26	1.4281	52	0.8192	11	55°.0
.1	0.5750	14	0.7028	26	1.4229	53	0.8181	10	54°.9
.2	0.5764	15	0.7054	26	1.4176	52	0.8171	10	.8
.3	0.5779	14	0.7080	27	1.4124	53	0.8161	10	.7
.4	. . .		. . .		. . .		. . .		.6
.5	. . .		. . .		. . .		. . .		.5
	. . .		. . .		. . .		. . .		
	N COS	d	N COT	d	N TAN	d	N SIN	d	ANGLE

* See *Logarithmic and Trigonometric Tables*, compiled by E. J. Wilczynski and H. E. Slaught.

PROBLEM 1. Find the functions of 35°.2.

Solution. In the left-hand column find 35°.2. The four numbers which are printed in the horizontal row to the *right* of 35°.2 are, *from left to right*, the sine, tangent, cotangent, and cosine of 35°.2, as indicated by the name printed at the *head* of each of these columns. Therefore

$$\sin 35°.2 = 0.5764, \tan 35°.2 = 0.7054, \cot 35°.2 = 1.4176,$$
$$\cos 35°.2 = 0.8171.$$

PROBLEM 2. Find the functions of 54°.8.

Solution. In the right-hand column find 54°.8. The four numbers which are printed in the horizontal row to the *left* of 54°.8 are, *from right to left*, the sine, tangent, cotangent, and cosine of 54°.8, as indicated by the name printed at the *foot* of each of these columns. Therefore

$$\sin 54°.8 = 0.8171, \tan 54°.8 = 1.4176, \cot 54°.8 = 0.7054,$$
$$\cos 54°.8 = 0.5764.$$

Thus every number of the table does double duty. For example, 0.5764 is both the sine of 35°.2 and the cosine of 54°.8, as it should be. (See Art. 10, equations (2).)

Angles less than 45° are given in the left-hand column of the table, and the names of the corresponding functions are found at the *top* of the page. Angles greater than 45° are given in the right-hand column with the names of the functions at the *bottom* of the page.

The table gives the values of the functions only for every tenth of a degree. If the given angle contains fractional parts of this unit, its functions cannot be read directly from the table. In such cases we make use of the process of **interpolation**, the nature of which will become apparent from the following examples.

PROBLEM 3. Find the sine of 35°.17.

Solution. This angle lies between 35°.1 and 35°.2. More precisely, it lies $\frac{7}{10}$ of the way from the former toward the latter. We conclude that its sine will be $\frac{7}{10}$ of the way from

$$\sin 35°.1 = 0.5750 \text{ toward } \sin 35°.2 = 0.5764.$$

But the difference d between these last two numbers is 0.0014, seven tenths of which is equal to 0.0010 (reduced to four decimal places). Therefore

$$\sin 35°.17 = 0.5750 + 0.0010 = 0.5760.$$

PROBLEM 4. Find the cotangent of 35°.17.

Solution. From the table we find

$$\begin{aligned} \cot 35^\circ.1 &= 1.4229 \\ \cot 35^\circ.2 &= \underline{1.4176} \\ d = \cot 35^\circ.2 - \cot 35^\circ.1 &= -0.0053 \qquad \text{(tabular difference)} \end{aligned}$$

We must add $\frac{7}{10}$ of d to cot 35°.1. But $\frac{7}{10}d = -0.0037$.

Therefore $$\cot 35^\circ.17 = 1.4192.$$

We observe that in problem 3 the correction was positive, while in problem 4 it was negative.

If we always interpolate from the smaller toward the larger angle, the correction will be positive in the case of sine and tangent, negative in the case of cosine and cotangent. For, the former two functions *increase* with the angle, while the latter two *decrease.*

There will never be any serious danger of giving the wrong sign to the correction, if we cultivate the habit of running through the numbers of the table near the place we are using, so as to see in which direction they are growing.

EXERCISE VI

1. Find all of the functions of 15°.3, 28°.7, 63°.4, 82°.1.

2. Find sin 37°.24, cos 62°.19, tan 53°.42, cot 27°.13.

3. Formulate in words the principle upon which the method of interpolation is based. Is this principle absolutely exact, or is it in the nature of an approximation?

4. What are the numbers in the four narrow columns of the table headed d, and what purpose do they serve?

5. In the arrangement of the table as explained, what use has been made of the results of Art. 10?

We have shown how to find the functions when the angle is given. It remains to show how to find an angle when one of its functions is known. The general method will be apparent from the following examples.

PROBLEM 5. The tangent of an unknown acute angle A is equal to 0.7046. Find the angle A.

Solution. In the specimen table on page 25 we observe that the number 0.7046 does not occur in the tangent column. However, we find there

the two numbers 0.7028 and 0.7054 between which 0.7046 lies. Thus we have

$$\tan 35^\circ.1 = 0.7028,$$
$$\tan A = 0.7046,$$
$$\tan 35^\circ.2 = 0.7054.$$

Between $\tan 35^\circ.1$ and $\tan A$, the difference is 0.0018.

Between $\tan 35^\circ.1$ and $\tan 35^\circ.2$, the difference is 0.0026.

Therefore, $\tan A$ is $\frac{18}{26}$ of the way from $\tan 35^\circ.1$ toward $\tan 35^\circ.2$, and consequently

$$A = 35^\circ.1 + \tfrac{18}{26} \text{ of one tenth of a degree,}$$

or

$$A = 35^\circ.1 + 0^\circ.07 = 35^\circ.17,$$

reducing to the nearest hundredth of a degree.

PROBLEM 6. Find the acute angle A whose cosine is 0.5772.

Solution. We have

$$\cos 54^\circ.7 = 0.5779,$$
$$\cos A = 0.5772,$$
$$\cos 54^\circ.8 = 0.5764,$$

whence

$$\cos A - \cos 54^\circ.7 = -0.0007,$$
$$\cos 54^\circ.8 - \cos 54^\circ.7 = -0.0015.$$

Therefore, $\cos A$ is $\frac{7}{15}$ of the way from $\cos 54^\circ.7$ toward $\cos 54^\circ.8$. Hence

$$A = 54^\circ.7 + \tfrac{7}{15} \text{ of one tenth of a degree}$$

or

$$A = 54^\circ.7 + 0^\circ.05 = 54^\circ.75.$$

With a little practice, the student will soon become sufficiently expert in the process of interpolation to enable him to perform this operation mentally. He should train himself with that end in view.

If we wish to find, by means of our table, the natural functions of an angle which is expressed in degrees, minutes, and seconds, the minutes and seconds should first be converted into decimal parts of a degree. This may easily be done, remembering that $1' = \frac{1}{60}$ of a degree and $1'' = \frac{1}{60}$ of a minute. Table VII may be used to save time in making this transformation.

EXERCISE IX

1. Find the values of

$$\sin 18^\circ\, 12',\ \cos 67^\circ\, 23',\ \tan 58^\circ\, 34',\ \cot 64^\circ\, 16'.$$

2. Find the acute angles for which

$$\sin A = 0.5673, \qquad \tan C = 1.7328,$$
$$\cos B = 0.2791, \qquad \cot D = 0.8924.$$

13. Solution of right triangles by means of the table of natural functions. A right triangle is determined by any two of its parts (not counting the right angle) provided that at least one of these parts is a side. The relations, which we have found between the angles and sides, enable us to compute the remaining parts of a right triangle when any two such parts are given. In order to make sure of this fact, let us see what cases may present themselves.

If both of the given parts are sides, there are two possibilities. Either the two given sides include the right angle (Case 1), or else one of the given sides is the hypotenuse (Case 2). If one of the given parts is a side and one is an angle, we have again two possibilities, viz.: given the hypotenuse and one acute angle (Case 3); or, given one leg and one acute angle (Case 4). We proceed to discuss these cases in order.

CASE 1. *Given the two sides of the triangle which include the right angle (the two legs).*

In our previous notation this means that a and b are given. In this case we may first compute

$$\tan A = \frac{a}{b}$$

and then find A from the table. We may then find c from

$$c = \frac{a}{\sin A} = \frac{b}{\cos A}$$

in two ways (affording a check), and B from

$$B = 90^\circ - A.$$

CASE 2. *Given one leg and the hypotenuse.*

We may denote the given leg by a. Then a and c are given. The solution is accomplished by means of the formulæ

$$\sin A = \frac{a}{c}, \quad b = a \cot A = c \cos A, \quad B = 90^\circ - A.$$

CASE 3. *Given the hypotenuse and one acute angle.*

We may denote the given acute angle by A. Then A and c are given. The solution is given by the equations

$$a = c \sin A, \quad b = c \cos A, \quad B = 90° - A.$$

CASE 4. *Given one leg and one acute angle.*

Since the knowledge of one acute angle implies that of the other, we may assume that a and A are the given parts. To find the remaining parts, we use the formulæ

$$b = a \cot A, \quad c = \frac{a}{\sin A} = \frac{b}{\cos A}, \quad B = 90° - A.$$

Our discussion has shown that the methods at our disposal suffice to find the remaining parts of a right triangle when two independent parts of the triangle (not counting the right angle) are given. To establish this fact was the purpose of the above classification. It is not necessary, nor even desirable, when solving a numerical problem of this sort, to find out first under what case it falls. In practice it is better not to refer to this classification at all, but to pick out and solve those among the four equations

$$(1) \qquad \sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b}, \quad A + B = 90°$$

which contain *only one unknown* quantity each. The remaining equations may then, in most cases, serve as a check.

A more complete check is given by the equation

$$(2) \qquad a^2 + b^2 = c^2.$$

In order to avoid the inconvenience of forming the quantities a^2, b^2, c^2 by actual multiplication, we have supplied a table of squares (Table VI). The arrangement and use of this table will be apparent from the following examples.

EXAMPLE 1. Find the squares of 0.324 and of 3.24.

Solution. In the left-hand column of Table VI we find 0.32. In the same horizontal row with this number, and in the column headed 4, we find 0.1050. Therefore

$$(0.324)^2 = 0.1050, \quad (3.24)^2 = 10.50.$$

EXAMPLE 2. Find the squares of 0.3243, of 3.243, and of 32.43.

Solution. From the table we find

$$(0.324)^2 = 0.1050, \quad (0.325)^2 = 0.1056.$$

The difference between the two squares is 0.0006. The number 0.3243 is three tenths of the way from 0.324 toward 0.325. Therefore, its square will be three tenths of the way from 0.1050 toward 0.1056. That is

$$(0.3243)^2 = 0.1050 + \tfrac{3}{10} \text{ of } 0.0006 = 0.1050 + 0.0002 = 0.1052,$$

and

$$(3.243)^2 = 10.52, \quad (32.43)^2 = 1052.$$

EXAMPLE 3. Find the square root of 0.5520.

Solution. We find from the table that this number is the square of 0.743. Therefore

$$\sqrt{0.5520} = 0.743.$$

EXAMPLE 4. Find the square root of 0.5525.

Solution. The table gives

$$\sqrt{0.5520} = 0.743, \quad \sqrt{0.5535} = 0.744.$$

Therefore, by interpolation

$$\sqrt{0.5525} = 0.743 + \tfrac{5}{15} \text{ of } 0.001 = 0.743 + 0.0003 = 0.7433.$$

In engineering practice, equation (2) is used very extensively in connection with such tables of squares, not merely for checking, but for the purpose of performing the original calculation. The tables of INSKIP and SMOLEY are particularly convenient for this purpose, if the distances are expressed in feet, inches, and thirty-seconds of an inch.

The following examples illustrate the methods for solving right triangles.

EXAMPLE 5. In a right triangle, right angled at C, given $a = 3.479$, $b = 2.321$. Compute the remaining parts of the triangle.

Solution. We find first

$$\tan A = \frac{a}{b} = \frac{3.479}{2.321} = 1.4989.$$

The table of tangents gives

$$A = 56°.2 + \tfrac{11}{12} \text{ of } 0°.1 = 56°.29,$$

reducing to the nearest hundredth of a degree, as usual. The tables of sines and cosines give

$$\sin A = 0.8320, \quad \cos A = 0.5548.$$

We compute next

$$c = \frac{a}{\sin A} = \frac{3.479}{0.8320} = 4.181,$$

and use the equation $c \cos A = b$ as a check. We find

$$c \cos A = 2.320, \quad b = 2.321.$$

The two members of the check equation do not agree exactly, but we have no right to expect absolute agreement. All of the numbers used in the calculation are merely approximations, giving us the values of the functions to the *nearest* unit of the fourth decimal place. In combining several such approximate numbers, the error may occasionally exceed two or three units of the last decimal place.

Finally we find

$$B = 90° - A = 33°.70.$$

We may exhibit this solution more compactly as follows. The figures in parentheses indicate the order in which the various results are obtained.

Formulæ. $\tan A = \frac{a}{b}, \; c = \frac{a}{\sin A}, \; B = 90° - A.$

Check. $c \cos A = b.$

$$\text{Given}\begin{cases} a = 3.479 & (1) \\ b = 2.321 & (2) \end{cases} \quad \begin{aligned} \sin A &= 0.8320 \; (5) \\ \cos A &= 0.5546 \; (6) \end{aligned} \quad \begin{aligned} c &= 4.181 \; (7) \\ c \cos A &= 2.320 \; (8) \text{ Check.} \end{aligned}$$

$$\tan A = 1.4989 \; (3) \qquad A = 56°.30 \; (4) \qquad B = 33°.70 \; (9).$$

EXAMPLE 2. In a right triangle, right angled at C, given $c = 5.783$, $A = 42°.39$. Compute the remaining parts of the triangle.

Solution. Formulæ. $a = c \sin A, \quad b = c \cos A, \quad B = 90° - A.$

Check. $a^2 + b^2 = c^2.$

$$\begin{aligned} \text{Given} \begin{cases} c = 5.783 & (1) \\ A = 42°.39 & (2) \end{cases} \\ B = 47°.61 \; (3) \\ \sin A = 0.6742 \; (4) \\ \cos A = 0.7386 \; (5) \end{aligned} \qquad \begin{aligned} a &= 3.899 \; (6) \\ b &= 4.271 \; (7) \\ a^2 &= 15.20 \; (8) \\ b^2 &= 18.24 \; (9) \\ \left.\begin{aligned} a^2 + b^2 &= 33.44 \; (10) \\ c^2 &= 33.44 \; (11) \end{aligned}\right\} &\text{ Check.} \end{aligned}$$

Remark. The quantities a^2, b^2, c^2 required for the check were obtained from the table of squares. When no such table is available, it is usually desirable to write the check equation in the form

$$a^2 = c^2 - b^2 = (c - b)(c + b),$$

since this form of the equation reduces by one the number of multiplications required.

In this example, the check computation would then yield

$$a^2 = 15.202, \qquad \begin{aligned} c - b &= 1.512, \\ c + b &= 10.054, \\ c^2 - b^2 &= 15.202 \quad \text{Check.} \end{aligned}$$

EXERCISE X

In each of the following right triangles, right angled at C, two parts are given. Compute the remaining parts and check. Also check by means of a graphic solution to provide against gross errors.

1. $a = 27, \quad A = 25°.1.$
2. $a = 34.5, \quad c = 52.8.$
3. $a = 2.781, \quad b = 3.056.$
4. $b = 87.95, \quad A = 55°.36.$
5. $c = 604.5, \quad A = 47°.53.$
6. $a = 8.695, \quad b = 7.321.$
7. $b = 62.78, \quad c = 81.39.$
8. $B = 29°.58, \quad c = 2354.$

9. A gravel roof slopes three fourths of an inch per horizontal foot. What angle does it make with a horizontal plane?

10. The pitch of a gable roof is the quotient obtained by dividing the height of the ridge-pole above the garret floor by $\frac{1}{2}$ the width of the floor. What is the pitch of a gable roof covering a garret 38 feet wide, if the ridge-pole is 15 feet above the garret floor, and what angle does the roof make with a horizontal plane?

11. At a time when the sun was 55° above the horizon, the shadow of a certain building was found to be 112 feet long. How high is the building?

12. The side of a regular decagon is 3.471 feet. Find the radii of the inscribed and circumscribed circles.

13. The side of a regular polygon of n sides is equal to a. Find formulæ for the radii of the inscribed and circumscribed circles.

CHAPTER IV

DISCUSSION OF SOME DEVICES FOR REDUCING THE LABOR INVOLVED IN NUMERICAL COMPUTATIONS

14. The number of decimal places. As we attempted to point out in Chapter I, every number obtained as a result of measurement is really an approximation. If we measure the distance between two dots on our drawing board, by means of a carefully constructed scale which reads to the fiftieth part of an inch, we may still estimate half of this smallest scale unit with the naked eye. Let us assume that the divisions of the scale are reliable, that the ruler is very nearly straight, that the dots are very small, and that we are using the greatest of care in our measurement. We may then concede that the result of such a measurement (say 5.34 inches) is accurate to the nearest $\frac{1}{100}$ of an inch. This means that no number, with two figures to the right of the decimal point, is as close to the true value as 5.34. It means that 5.33 is certainly too small and that 5.35 is certainly too large. It does not mean that the true value is exactly 5.34 inches, but that the true value lies between 5.335 and 5.345 inches.

When we record the result of such a measurement, the number of decimal places which we write (two in this example) is an indication of the degree of precision which we claim for the result. In this connection, let us note emphatically that a zero, when obtained as the last digit of the measure of a quantity, should never be suppressed. Suppose, for instance, that in the above example we had obtained 5.30 inches as the result of our measurement. This means that we are certain that the true value of the distance lies somewhere between 5.295 and 5.305 inches.

If we were to record this result as 5.3 inches, suppressing the final zero, we should be giving the erroneous impression that we had measured the distance only to the nearest tenth of an inch and that it might have any value between 5.25 and 5.35 inches.

Thus, *the number of decimal places, which we use in recording the result of a measurement, is an indication of the degree of precision which we attribute to this result.*

This being so, we are guilty of negligence whenever we express such a result by a decimal with fewer places than we are able to guarantee. For we are thus throwing away knowledge which we have actually had in our possession. But it would be dishonest to express our result with *more* decimal places than we can guarantee. For we should then be tending to mislead others into thinking our measurements more accurate than they really are.

We may estimate the degree of precision of a number, as we have just done, on an *absolute* scale. But clearly it will usually be more reasonable to adopt as a measure of precision the ratio of the "probable error" of the measurement to the total magnitude of the quantity measured. If we do this, an error of one foot in a thousand is to be regarded as of no greater importance than an error of $\frac{1}{1000}$ of an inch in an inch. In either case we may say that it is an error of $\frac{1}{10}$ of one per cent.

Whenever we properly record the result of a measurement by a number consisting of four digits, no matter where the decimal point may be placed, this means, in the light of our preceding discussion, that the error of the last digit is guaranteed to be less than half a unit of the last decimal place. Now, the smallest number expressed by four digits is 1000. Let us suppose that the exact value of our unknown quantity is $x = 1000$ units, but that as a result of our measurement we have found $x = 999.5$ units. Then our error is $\frac{1}{2000}$, or $\frac{1}{20}$ of one per cent, of the total magnitude measured. The largest number expressible by four digits is 9999. If

the exact value of x is 9999 and our error of measurement is half a unit, the error will be to the total magnitude measured as $\frac{1}{2}$ is to 9999, or as 1 is to 19998. It will be an error of about $\frac{1}{200}$ of one per cent of the whole. Thus, four digits are certainly sufficient to express the result of a measurement if its degree of accuracy does not exceed $\frac{1}{20}$ of one per cent. Now, most of the ordinary operations of surveying fall well within this limit, so that four decimal places are usually sufficient to express the results obtained by the surveyor.

15. The accuracy of a sum, difference, product, or quotient of two numbers obtained by measurement. It is clear that the sum or difference of two numbers can have no greater precision than the less accurate of the two numbers. Consequently, it is useless and misleading to retain more decimal places in one term of a sum or difference than in the others.

In forming a product we are apt to do a great deal of useless work if we fail to remember that the factors, and therefore the product, are mere approximations. Suppose we have measured the sides a and b of a rectangular field to the nearest hundredth of a foot and have found $a = 35.67$ ft., $b = 86.72$ ft. To find the area of the field we form the product ab. The ordinary method (shown in the margin) gives the result 3093.3024 square feet. But if we allow the result to stand in this form, we shall exhibit either our ignorance or our desire to create a false impression. For this would seem to indicate a result precise to the nearest $\frac{1}{10000}$ of a square foot, whereas it is uncertain by more than a whole square foot as we shall now show.

$$\begin{array}{r} 86.72 \\ 35.67 \\ \hline 60704 \\ 52032 \\ 43360 \\ 26016 \\ \hline 3093.3024 \end{array}$$

In fact, the equations $a = 35.67$ ft., $b = 86.72$ ft. merely mean that a and b are between the limits

$$35.665 < a < 35.675, \quad 86.715 < b < 86.725$$

respectively, so that the area must be between the limits

$$86.715 \times 35.665 < ab < 86.725 \times 35.675$$

or

$$3092.685475 < ab < 3093.919375.$$

The uncertainty in the value of ab is therefore so great that no digit to the right of the decimal point has any real significance. Even the last digit preceding the decimal point is not certain, so that we are even slightly overstating our accuracy, if we write simply

$$ab = 3093 \text{ square feet.}$$

Thus, we see that the product of two approximate four-place numbers is to be regarded as accurate to no more than four places. Consequently, it is a waste of time and labor to actually work out all of the partial products in the multiplication. We might abbreviate the process as follows:

86.72	or better, by	86.72
35.67	writing the	35.67
6.	more important	2602.
52.	partial products	434.
434.	first;	52.
2602.		6.
3094.		3094.

The fact that we find 3094, instead of 3093, need not disturb us, since we have seen from our above considerations that the last digit is actually uncertain to the extent of one unit.

This process, which is sometimes called *abbreviated multiplication*, is obviously preferable to the ordinary method, since it does away with the labor of first finding numbers which must afterwards be discarded.

Similar remarks may be made for division. More generally, whenever we are dealing with numbers whose first four digits only can be regarded as certain, it is wise to abbreviate all of our calculations correspondingly. It should be remarked, however, that, in certain exceptional cases, very exact results may be obtained from inaccurate data and *vice versa*. But this is not the place for a discussion of such cases.

16. Labor-saving devices. We have seen that extensive calculations, even with four-place numbers, are apt to be

troublesome and laborious. In many problems it is necessary to use five- six- or seven-place numbers. In such cases, the amount of labor required becomes excessive, even if we make use of the abbreviated method of multiplication and division.

A very important aid in performing such calculations is furnished by certain tables, such as tables of squares (of which Table VI is an example), tables of cubes, and reciprocals of numbers, etc. CRELLE's Tables, which are merely a systematically arranged and extensive set of multiplication tables, are particularly valuable.

A second great aid to numerical calculation is furnished by calculating machines, which are now being used very extensively in commercial as well as scientific work. The ordinary cash register is one of the simplest of these machines.

Graphic methods for solving numerical problems constitute a third great class of aids to calculation. These methods have, in recent times, been modified and extended, so as to become capable of greater accuracy, and for many problems no other method of solution is known. The slide rule, which may be classed either with the graphic methods or among the calculating machines, has become so popular among engineers as to exclude, in their work, almost all other methods of calculation. (See Arts. 29 and 30.)

But the most important of all of these labor-saving devices, without which the slide rule and many similar contrivances cannot even be thoroughly understood, is the method of calculation by logarithms.

17. Definition of logarithms. It is apparent, from what has been said, that the real cause of the laboriousness of extensive calculations lies in the operations of multiplication and division. Addition and subtraction, even of numbers with many places, are comparatively easy. It was this fact which caused JOHN NAPIER (1550–1617) and JOBST BÜRGI

(1552–1632)* to consider the possibility of devising a method, by means of which addition and subtraction might be made to do the work of multiplication and division. The method which they invented for this purpose is essentially equivalent to the one which we shall now explain, though very different from it in form. We must remember that the notations of modern algebra, to which we are accustomed and which are of the greatest assistance to us in our mathematical arguments, are the results of a century-long process of development. This process was far from complete in Napier's and Bürgi's time. The greatness of the achievement of these men can only be properly appreciated when judged from the standpoint of the mathematical knowledge of those days.†

From our present point of view the possibility of reducing the operations of multiplication to that of addition is an immediate consequence of a familiar fact of algebra. This fact, embodied in the formula

$$a^x a^y = a^{x+y},$$

states that the product of two powers of the same base is itself a power of that base, whose exponent is equal to the sum of the exponents of the two original powers. We may express this fact by the statement that to the *multiplication* of two powers of the same base corresponds the *addition* of their exponents. By this simple remark, multiplication is actually converted into addition for all such numbers as are known powers of a common base.

We shall assume that the fixed number a, the base of our system, is positive and different from unity. If the exponent x is a positive integer, there will be comparatively few

* NAPIER was of Scotch and BÜRGI of Swiss nationality. Bürgi's discovery of logarithms was unquestionably independent of Napier's and was made at about the same time. But Napier's book "*Mirifici Logarithmorum canonis descriptio*," containing an account of his method was published in 1614, six years earlier than Bürgi's "*Arithmetische und Geometrische Progress-Tabuln*."

† For an excellent account of the history of logarithms see CAJORI in *The American Mathematical Monthly*, Vol. 20 (1913).

numbers which can be regarded as powers of a. Suppose, for example, that $a = 2$. Then 2, 4, 8, 16, 32, etc., are integral powers of 2, but 1, 3, 5, 6, 7, 9, 10, etc., are not. But, from our previous study of algebra, we are acquainted with the fact that the symbol a^x may be defined, not merely for the case when the exponent is a positive integer, but also when the exponent is any positive or negative rational number of the form $\pm\frac{p}{q}$ (p and q being integers), or zero. These definitions are as follows:

If x is a positive integer ($x = p$),

I. $a^x = a^p = a \cdot a \cdot a \cdots$ (a product of p factors each equal to a).

If x is a positive rational fraction $\left(x = \frac{p}{q}\right)$,

II. $$a^x = a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p,$$

where $\sqrt[q]{a}$ means the positive qth root of a.

If x is a negative rational fraction $\left(x = -\frac{p}{q}\right)$,

III. $$a^x = a^{-p/q} = \frac{1}{a^{p/q}} = \frac{1}{\sqrt[q]{a^p}}.$$

Finally, if $x = 0$,

IV. $$a^x = a^0 = 1.$$

Now, there is nothing remarkable about the fact that we have been able to define the symbol a^x in all of these cases. We might have done that in many different ways. But it *is* remarkable that, if we adopt these particular definitions, the formulæ

(1) $$a^x \cdot a^y = a^{x+y},$$

(2) $$\frac{a^x}{a^y} = a^{x-y},$$

(3) $$(a^x)^y = a^{xy}$$

turn out to be true, not only when the exponents are positive integers but in all of the four cases for which we have

defined the symbol a^x. That this should be so, is of course not merely a fortunate coincidence. It is due to the fact that the definitions II, III, IV were deliberately chosen in such a way as to insure the universal validity of formulæ (1), (2), and (3). These formulæ, which are collectively known as the three *index laws*, are fundamental for the following discussion.

Equations I, II, III, IV suffice to define the symbol a^x whenever x is a rational number. Now, every irrational number can be approximated, as closely as we may desire, by means of a decimal fraction; and this decimal fraction (which is a rational number) will take the place of the original irrational number in all numerical calculations. We may define a^x, when x is irrational, as follows:

Let x_1 be the closest approximation, to the irrational number x, which is possible by means of a decimal fraction with only one figure to the right of the decimal point. Let x_2 be the closest approximation possible by means of a decimal fraction with only two figures to the right of the decimal point. Let $x_3, x_4, \cdots x_n, \cdots$ be similar approximations with $3, 4, \cdots n$ figures after the decimal point. The sequence of rational numbers

$$x_1, x_2, x_3, \cdots x_n, \cdots$$

has the irrational number x as a limit. Then a^x is defined to be the limit of the second sequence of numbers

$$a^{x_1}, a^{x_2}, a^{x_3}, \cdots a^{x_n}, \cdots .^*$$

As an example, consider $a = 10$, $x = \sqrt{2}$. We have

$$x_1 = 1.4,\ x_2 = 1.41,\ x_3 = 1.414,\ x_4 = 1.4142, \text{ etc.},$$

$$a^{x_1} = 10^{1.4},\ a^{x_2} = 10^{1.41},\ a^{x_3} = 10^{1.414},\ a^{x_4} = 10^{1.4142}, \text{ etc.}$$

By $10^{\sqrt{2}}$ we mean the limit approached by the numbers of this second sequence.

For practical purposes, however, $\sqrt{2}$ is replaced by one of the approximations 1.4, 1.41, 1.414, etc., namely, the first one which is sufficiently accurate for the particular problem

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* These limits exist, but it would carry us too far to prove this fact.

under consideration; and $10^{\sqrt{2}}$ is replaced by the first one of the numbers $10^{1.4}$, $10^{1.41}$, etc., which is sufficiently close to the true value for the purposes of the problem in question.

It may be shown that, if the above definition of a^x for irrational values of x be adopted, *the index laws will hold also for irrational exponents.*

We may now state, without formal proof, a theorem which is fundamental in the theory of logarithms, in so far as the very existence of logarithms depends upon it. This theorem is as follows:

If a is a positive number different from unity, there exists one and only one exponent x (positive, negative, or zero), such that

$$a^x = N,$$

where N is any positive number.

Although we state this theorem without proof, the student may easily convince himself of its great plausibility by a process which, if carried out to its logical conclusion, would constitute a proof. Suppose, for instance, that $a = 2$ and that $N = 1000$. We have

$$2^5 = 32,\ 2^6 = 64,\ 2^7 = 128,\ 2^8 = 256,\ 2^9 = 512,\ 2^{10} = 1024.$$

We conclude that the value of x for which

$$2^x = 1000$$

must be between 9 and 10. Now $2^{1/2} = \sqrt{2} = 1.4142 \cdots$. Therefore

$$2^{9.5} = 2^9 . 2^{1/2} = 512 \times 1.4142 = 724.1.$$

But

$$2^{10} = 1024,$$

so that x must lie between 9.5 and 10. We may obtain closer and closer limits between which x must lie by continuing this process, and thus ultimately show that there exists a number x (as a limit of a sequence), for which $2^x = 1000$. This argument at the same time indicates a process by means of which the exponent x may be calculated to any desired number of decimal places.

We are now ready to define a logarithm.

The **logarithm** *of any positive number* N, *with respect to the base* a, *is the exponent of the power to which the base* a *must be raised in order to obtain the number* N.*

In other words, if $a^x = N$,

we say that x (the exponent) is the logarithm of N with respect to the base a. In symbols we write this same statement as follows: $x = \log_a N$.

Example. Since $5^3 = 125$, we have $\log_5 125 = 3$.

EXERCISE XI

1. What are the logarithms of 2, 4, 8, 16, 32, 64, 128 with respect to the base 2? Write out each of these results in symbols; thus, $\log_2 4 = 2$.

2. What are the logarithms of 3, 9, 27, 81, 243 with respect to the base 3?

3. What are the logarithms of 10, 100, 1000, 10,000 with respect to the base 10?

4. What are the logarithms of 3, 9, 27, 81, 243 with respect to the base 27?

5. What are the logarithms of 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, $\frac{1}{243}$ with respect to the base 3?

6. What are the values of 2^x, 3^x, 4^x, 10^x when x is equal to zero? What, then, is the logarithm of 1 with respect to each of the bases 2, 3, 4, 10?

7. What is the logarithm of 1 with respect to any base a?

8. What is the logarithm of any number with respect to itself as base?

9. Find, approximately to two decimal places, the number whose logarithm, with respect to the base 2, is equal to 1.5.

18. The properties of logarithms. Those properties of logarithms which are of importance for the purposes of numerical calculation, are immediate consequences of the index laws and of the definition of logarithms. In fact, we

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* The word logarithm is derived from the Greek λόγος or *logos*, meaning proportion or ratio, and ἀριθμός or *arithmos*, meaning number. The reason for choosing this name will be apparent from the theorem stated in Exercise XII, Ex. 4.

may write each of two positive numbers, M and N, in the form

$$(1) \qquad M = a^x, \quad N = a^y,$$

so that, in accordance with the definition of logarithms,

$$(2) \qquad x = \log_a M, \quad y = \log_a N.$$

According to the first index law (Art. 17, equation (1)), the product of M and N is equal to

$$MN = a^x \cdot a^y = a^{x+y},$$

whence, by the definition of logarithms,

$$\log_a (MN) = x + y = \log_a M + \log_a N.$$

The theorem expressed by this formula may obviously be extended to any number of factors. Hence,

I. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

From (1) we obtain by division,

$$\frac{M}{N} = a^{x-y},$$

making use of the second index law (Art. 17, equation (2)). Therefore, by the definition of logarithms,

$$\log_a \left(\frac{M}{N}\right) = x - y = \log_a M - \log_a N,$$

a result which may be formulated as follows :

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

The same fact may, of course, be stated in the equivalent form: *the logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.*

According to the third index law (Art. 17, equation (3)), we have

$$(a^x)^p = a^{px}.$$

Therefore, we find from (1)

$$M^p = a^{px},$$

or, by the definition of logarithms,

$$\log_a M^p = px = p \log_a M.$$

Consequently,

III. *The logarithm of the p^{th} power of a number M is obtained by multiplying the logarithm of M by p.*

Since the third index law is true whether p be an integer or a fraction, the last theorem has the following corollary, obtained by putting p equal to $\frac{1}{n}$:

IV. *The logarithm of the n^{th} root of a number M is obtained by dividing the logarithm of M by n.*

The content of the two equations

$$a^1 = a, \quad a^0 = 1$$

may be stated as follows:

V. *The logarithm of any number, with respect to itself as base, is equal to unity.*

VI. *The logarithm of unity, with respect to any base, is equal to zero.*

It is clear now how logarithms will serve to reduce the operations of multiplication and division to those of addition and subtraction. Suppose that we have at our disposal a *table* of logarithms. To multiply M by N we look up the logarithms of these numbers from the table and add them together. We then find the number whose logarithm is equal to this sum by again referring to the table; this number is the product MN. To divide M by N we proceed in the same way, except that in this case we form the difference $\log M - \log N$ instead of the sum.

EXERCISE XII

1. If $\log_{10} 2 = 0.30103$, $\log_{10} 3 = 0.47712$, find $\log_{10} 12$, $\log_{10} \left(\frac{3}{2}\right)$, $\log_{10} \left(\frac{27}{4}\right)$, $\log_{10} \sqrt[3]{6}$.

2. Express, in terms of $\log_a p$ and $\log_a q$, the following quantities:

$$\log_a (p^2 q^3),\quad \log_a \left(\frac{p^3}{q^2}\right),\quad \log_a \sqrt{\frac{p^{-4}}{q^{-7}}}.$$

3. Prove the truth of the following statement. If $\log_{10} x$ is expressed as a decimal fraction (x being a positive number greater than unity), the logarithm of $10^k x$ (k being a positive integer) will differ from $\log_{10} x$ only in its integral part.

4. Prove the theorem: If the numbers a_1, a_2, a_3, ... are in geometrical progression, their logarithms are in arithmetical progression.

5. Prove the equation

$$\log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a (x + \sqrt{x^2 - 1}).$$

CHAPTER V

CALCULATION BY LOGARITHMS

19. Common logarithms. With scarcely an exception, the civilized nations of all times have made use of the decimal system for expressing numbers, both in the spoken and in the written language.* For this reason, the number 10 is especially well adapted to serve as base for a system of logarithms. Logarithms with respect to the base 10 are usually known as *common logarithms;* they are also sometimes called *Briggsian* logarithms, in honor of HENRY BRIGGS (1556–1630),† who constructed the first table of common logarithms.

In this book we shall have very little occasion, hereafter, to speak of any except the common logarithms. We shall therefore agree to abbreviate the symbol $\log_{10} N$ to $\log N$, the base 10 being understood when no other base is mentioned explicitly.

The positive integral powers of 10, such as 10, 100, 1000, etc., the negative integral powers of 10, such as 0.1, 0.01, 0.001, etc., and the zero power of 10, which is equal to 1, are the only numbers whose common logarithms are integers. The logarithms of all other numbers have an integral and a fractional part.

The fractional part of the logarithm is called the **mantissa,** *while the integral part of the logarithm is known as its* **characteristic.**

* It is usually admitted that the predominance of the decimal system over all others is due to the fact that the normal human being has ten fingers. This opinion has certainly been generally held since the time of ARISTOTLE.

† BRIGGS was the first Savilian Professor of Geometry at Oxford. According to BALL (see Ball's *Primer of the History of Mathematics*), Briggs was also the first to make systematic use of the decimal notation in working with fractions.

20. Properties of the mantissa. We consider the mantissa and the characteristic separately because, in practice, the method for finding the characteristic of a logarithm is entirely different from that employed for finding its mantissa. The reason for this will appear from the following discussion.

Let us consider an example. From the definition of a common logarithm, we know that

(1) $\log \sqrt[4]{10} = \log 10^{\frac{1}{4}} = \frac{1}{4} \log 10 = \frac{1}{4} \cdot 1 = 0.25000.$

Now it is not difficult to compute $\sqrt[4]{10}$ by elementary methods. We may, for instance, first compute $\sqrt{10}$ to as many decimal places as we desire, and then extract the square root of the result. We find, in this way, to five decimal places,

(2) $$\sqrt[4]{10} = 1.77828$$

or, if we combine (1) and (2),

(3) $$\log 1.77828 = 0.25000.$$

From the theorem about the logarithm of a product, we conclude

$$\begin{aligned}\log 17.7828 &= \log (1.77828 \times 10) = \log 1.77828 + \log 10\\ &= 0.25000 + 1 = 1.25000,\\ \log 177.828 &= \log (1.77828 \times 100) = \log 1.77828 + \log 100\\ &= 0.25000 + 2 = 2.25000,\end{aligned}$$

. . . .

We observe that the numbers 1.77828, 17.7828, 177.828, etc., contain the same succession of digits, and differ from each other only in the position of the decimal point. Their logarithms, on the other hand, whose values we have just calculated, differ from each other only in the value of the characteristic.

Again, if we make use of the theorem about the logarithm of a quotient, we find from (3)

$$\log 0.177828 = \log \frac{1.77828}{10} = 0.25000 - 1,$$

$$\log 0.0177828 = \log \frac{1.77828}{100} = 0.25000 - 2,$$

. . . .

Now the negative quantities, which appear on the right members of these equations, are not written in the form which we ordinarily use for negative quantities. Thus, for instance, we have found the value of log 0.0177828 to be $0.25000 - 2$, a result which we should ordinarily write in the form -1.75000, to which it is obviously equal. If we agree to write every negative logarithm in this unusual form, as a difference between a *positive proper fraction* and an integer, thus making its fractional part positive, we gain the advantage that the mantissas will be the same for any two numbers which contain the same succession of digits, even if none of these digits appear to the left of the decimal point. We avoid, in this way, the necessity of using two different tables of mantissas, one for numbers greater, and one for numbers less, than unity.

Let us recapitulate the result of our discussion in two formal statements.

I. *We agree to express the logarithm of any positive number N in such a form that its mantissa shall be positive.*

This can be done whether log N is positive or negative, that is, whether N be greater or less than unity. In the latter case, the negativeness of log N is brought about entirely by means of the negative characteristic.

As a consequence of this agreement, the following statement will be true in all cases.

II. *If two numbers contain the same succession of digits, that is, if they differ only in the position of the decimal point, their logarithms will have the same mantissa and will differ only in the value of the characteristic.*

It is for this reason that the tables give only the *mantissas* of the logarithms and that, in looking up the mantissas, we pay no attention to the position of the decimal point in the given number.

21. Determination of the characteristic. The characteristic of a logarithm is easily determined by inspection. Its value

depends merely on the position of the decimal point. Since we have

$$10^0 = 1,\ 10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \text{ etc.,}$$

or

$$\log 1 = 0,\ \log 10 = 1,\ \log 100 = 2,\ \log 1000 = 3, \text{ etc.,}$$

we draw the following conclusions:

If $1 < N < 10$, then $0 < \log N < 1$. $\therefore \log N$ has the characteristic 0.

If $10 < N < 100$, then $1 < \log N < 2$. $\therefore \log N$ has the characteristic 1.

If $100 < N < 1000$, then $2 < \log N < 3$. $\therefore \log N$ has the characteristic 2.

. . . .

If $10^k < N < 10^{k+1}$, then $k < \log N < k + 1$. $\therefore \log N$ has the characteristic k.

We may formulate these results as follows:

III. *If k is a positive integer, and if the number N lies between 10^k and 10^{k+1}, the characteristic of log N is equal to k.*

Since such a number N has $k + 1$ digits to the left of the decimal point, we obtain the following rule:

IV. *If N is any number greater than* 1, *the characteristic of its logarithm is one less than the number of digits in its integral part.*

The student is advised to make but little use of this rule on account of its mechanical character. Statement III provides a better method (less mechanical and easier to remember) for determining the characteristic.

It remains to show how to find the characteristic of log N when $N < 1$.

If $.1 < N < 1$, then $-1 < \log N < 0$. $\therefore \log N$ has the characteristic -1.

If $.01 < N < .1$, then $-2 < \log N < -1$. $\therefore \log N$ has the characteristic -2.

If $.001 < N < .01$, then $-3 < \log N < -2$. $\therefore \log N$ has the characteristic -3.

. . . .

If $\frac{1}{10^{k+1}} < N < \frac{1}{10^k}$, then $-(k+1) < \log N < -k$. $\therefore \log N$ has the characteristic $-(k+1)$.

Examination of this table leads to the following two statements, either of which may be used to determine the characteristic of $\log N$ when $N < 1$.

If k is a positive integer, and if the number N lies between $\frac{1}{10^k}$ and $\frac{1}{10^{k+1}}$, *the characteristic of $\log N$ is $-(k+1)$.*

If N is less than 1, *and is expressed as a decimal fraction having k zeros between the decimal point and the first significant figure, then the characteristic of the logarithm of N is $-(k+1)$.*

In one of our illustrations we had found

$$\log 0.0177828 = 0.25000 - 2.$$

We must never write this in the form

$$\log 0.0177828 = -2.25000,$$

since only the characteristic is negative and not the fractional part. Some computers use the notation

$$\log 0.0177828 = \bar{2}.25000;$$

but for most purposes it is preferable to write

$$\log 0.0177828 = 8.25000 - 10,$$

and similarly

$$\log 0.177828 = 9.25000 - 10.$$

In other words, *in actual practice, we write a positive characteristic* $10 - k$ *in place of the negative characteristic* $-k$, *and then subtract* 10 *from the whole logarithm.*

22. Arrangement and use of the table of logarithms. We have already mentioned the fact that the table of logarithms gives only the mantissas. The characteristics must be supplied by the computer by the methods of Art 21.

The table which we shall ordinarily use (Table I) gives the mantissa, for every number from 1 to 9999, to five decimal places.

In order to explain the arrangement of this table, we shall reprint a small portion of it, and solve a number of typical examples, chosen in such a way as to require only this part of the table for their solution.

N	0	1	2	3	4	5	6	7	8	9	PP		
												19	20
—	—	—	—	—	—	—	—	—	—	—	1	1.9	2.0
—	—	—	—	—	—	—	—	—	—	—	2	3.8	4.0
—	—	—	—	—	—	—	—	—	—	—	3	5.7	6.0
220	34242	262	282	301	321	341	361	380	400	420	4	7.6	8.0
221	439	459	479	498	518	537	557	577	596	616	5	9.5	10.0
222	635	655	674	694	713	733	753	772	792	811	6	11.4	12.0
223	830	850	869	889	908	928	947	967	986	*005	7	13.3	14.0
—	—	—	—	—	—	—	—	—	—	—	8	15.2	16.0
—	—	—	—	—	—	—	—	—	—	—	9	17.1	18.0
—	—	—	—	—	—	—	—	—	—	—			

PROBLEM 1. Find the logarithm of 221.4.

Solution. To find the mantissa we ignore the decimal point. We read down the left-hand column of the table (headed N) until we find the first three digits of our number, viz., 221. The numbers printed in the same horizontal row with 221 are, in order, the mantissas of the logarithms of 2210, 2211, 2212, ..., 2219, as indicated by the number at the head of each of the next ten columns. To save space, however, the first two digits of the mantissa are never printed more than once in each row. In our case we find the mantissa, from the column headed 4, to be .34518. Since 221.4 is between $100 = 10^2$ and $1000 = 10^3$, the characteristic is 2. Therefore

$$\log 221.4 = 2.34518.$$

PROBLEM 2. Find log 22.39.

Solution. Looking for the mantissa as before, we find *005. The asterisk indicates that the first two digits of the mantissa are not 34, as one might suppose, but 35. The reason for this appears clearly from the table. Therefore

$$\log 22.39 = 1.35005.$$

If the number N contains more than four digits, its logarithm cannot be read directly from the table. But it may be found by *interpolation.* We illustrate this process by an example.

PROBLEM 3. Find log 222.73.

Solution. From the table we find, supplying the characteristics ourselves,

$$\begin{aligned}\log 222.70 &= 2.34772\\ \log 222.80 &= 2.34792\end{aligned}$$

Tabular difference $= 0.00020 = 20$ units of the fifth decimal place.

Since 222.73 is $\frac{3}{10}$ of the way from 222.70 toward 222.80, we add $\frac{3}{10}$ of the tabular difference to log 222.70. Therefore

$$\log 222.73 = 2.34772 + \tfrac{3}{10} \text{ of } 0.00020,$$

or

$$\log 222.73 = 2.34772 + 0.00006 = 2.34778.$$

The auxiliary tables in the margin, headed P P (abbreviation for proportional parts), facilitate the process of interpolation.

Thus, in problem 3, we refer to the auxiliary table with 20 (the tabular difference) at its head. In the third row we find $\frac{3}{10}$ of 20 or 6.0.

It remains to show how to find the number when its logarithm is given.

PROBLEM 4. Given $\log N = 9.34857 - 10$. Find the value of N to five significant figures.

Solution. The characteristic of $\log N$ is $9 - 10$ or -1. Therefore, the number N must be between $10^{-1} = 0.1$ and $10^{0} = 1$. Consequently, the decimal point will precede the first significant figure of N.

The mantissa 34857 does not occur in the table, but it falls between the two tabular mantissas 34850 and 34869.

Thus we have:

$$9.34850 - 10 = \log 0.22310 \text{ (from the table)},$$

$$9.34857 - 10 = \log N,$$

$$9.34869 - 10 = \log 0.22320 \text{ (from the table)},$$

so that N lies between 0.22310 and 0.22320.

We observe that $\log N$ lies $\frac{7}{19}$ of the way from $\log 0.22310$ toward $\log 0.22320$. Therefore, N lies $\frac{7}{19}$ of the way from 0.22310 toward 0.22320. That is,

$$N = 0.22310 + \tfrac{7}{19} \text{ of 10 units of the fifth decimal place.}$$

But

$$\tfrac{7}{19} \text{ of 10 units} = \tfrac{70}{19} \text{ units} = 3\tfrac{13}{19} \text{ units} = 4 \text{ units.}$$

Therefore

$$N = 0.22310 + 0.00004 = 0.22314.$$

Also in this inverse problem (to find the number when its logarithm is given) interpolation is aided by the auxiliary tables in the margin.

Thus, in problem 4, the tabular difference is 19. The difference between $\log N$ and the smaller of the two tabular logarithms, between which $\log N$ lies, is 7. The auxiliary table with 19 at its head, shows that, among the tenths of 19, that one, which comes closest to the value 7, is the fourth. Consequently, N is $\frac{4}{10}$ of the way from 0.22310 toward 0.22320. Therefore, up to five decimal places, $N = 0.22310 + 0.00004 = 0.22314$.

23. Cologarithms. Since we obtain the same result whether we divide N by M, or multiply N by $\frac{1}{M}$, we may, in a logarithmic calculation, add the logarithm of $\frac{1}{M}$ instead of subtracting $\log M$. *The logarithm of* $\frac{1}{M}$ *is called the* **cologarithm** *of* M. Therefore

$$\operatorname{colog} M = \log \frac{1}{M} = \log 1 - \log M = -\log M,$$

since log 1 is equal to zero.

Cologarithms, like logarithms, are written with positive mantissas. Consequently, the cologarithm of a number is most easily found by subtracting its logarithm from zero, written in the form $10.00000 - 10$, as in the following example.

Problem 5. Find the cologarithm of 222.73.

Solution.

$$\begin{array}{r} 10.00000 - 10 \\ \underline{\log 222.73 = 2.34778 } \\ \operatorname{colog} 222.73 = 7.65222 - 10 \end{array}$$

It is easy to perform this operation of subtraction from $10.00000 - 10$ mentally. There is no gain, however, from the use of cologarithms when we are dealing with a quotient of only two numbers. A real advantage is gained by the introduction of cologarithms, when more than two logarithms are to be combined by addition and subtraction. For the logarithms which are to be subtracted we then substitute

cologarithms, enabling us to complete the operation by a single addition.

It often happens, just as in the case of forming a cologarithm, that we wish to subtract a logarithm from another smaller one. In all such cases we change the form of the minuend by adding and subtracting 10, or some convenient multiple of 10, as in the following example.

PROBLEM 6. Compute $\frac{32.34}{472.3}$.

Solution. We find from Table I,

$$\log 32.34 = 1.50974,$$
$$\log 472.3 = 2.67422.$$

In order to subtract the latter logarithm from the former, we write

$$\begin{array}{l} \log 32.34 = 11.50974 - 10,^* \\ \underline{\log 472.3 = 2.67422\qquad\qquad} \\ \log \frac{32.34}{472.3} = 8.83552 - 10 \end{array}$$

Hence, from the table, $\frac{32.34}{472.3} = 0.068473.$

24. Extraction of roots by means of logarithms. Since

$$\log \sqrt[p]{x} = \log x^{1/p} = \frac{1}{p} \log x,$$

it is easy to extract roots of any order by means of logarithms. If the characteristic of $\log x$ is not negative, no further remark is necessary. If $\log x$ is negative, we proceed as in the following example:

PROBLEM 7. Compute by logarithms: $\sqrt{.53760}$, $\sqrt[3]{.53760}$, and $\sqrt[5]{.53760}$.

Solution. $\log 0.53760 = 9.73046 - 10.$

$$\log \sqrt{.53760} = \tfrac{1}{2} \log 0.53760 = \tfrac{1}{2}(19.73046 - 20) = 9.86523 - 10.$$
$$\log \sqrt[3]{.53760} = \tfrac{1}{3} \log 0.53760 = \tfrac{1}{3}(29.73046 - 30) = 9.91015 - 10.$$
$$\log \sqrt[5]{.53760} = \tfrac{1}{5} \log 0.53760 = \tfrac{1}{5}(49.73046 - 50) = 9.94609 - 10.$$

Therefore, from Table I,

$$\sqrt{.53760} = .73322,\quad \sqrt[3]{.53760} = .81312,\quad \sqrt[5]{.53760} = .88326.$$

* A computer with some experience will refrain from actually writing the logarithm in the form 11.50974 – 10. It is easy for him to carry out the calculation *as though* it were so written.

25. Logarithmic calculations which involve negative numbers. We have only defined the logarithms of positive numbers. But this suffices for our purposes. Clearly, when we compute a product or quotient, its numerical value may be found first, without paying any attention to the signs of the various factors. Afterwards, the proper sign (+ or −) may be prefixed to the result according as there is an even or an odd number of negative factors.

The easiest way to keep a count of the negative factors is to use the method, introduced by GAUSS,* of writing the letter n immediately after a logarithm which corresponds to a negative number. In forming a sum or difference of logarithms, we write an n after the result only if an *odd* number of the separate logarithms is affected by an n.

EXAMPLE. If $N = -222.73$, we write

$$\log N = 2.34778\ n.$$

EXERCISE XIII

1. Making use of the tables, find log 3726, log 67.43, log 729800, log 0.3896, log 0.008527.

2. Making use of the tables, find log 32653, log 76.431, log 879450, log 0.045723, log 0.0059426.

3. By means of the tables, find the numbers whose logarithms are 3.84522, 1.68079, 8.89064 − 10, 7.12548 − 10, 2.27068.

4. By means of the tables, find the numbers whose logarithms are 3.89067, 9.24110 − 10, 1.52195.

5. Given $a = 3.1572$, $b = 7.2916$, $c = 45.731$. Compute by logarithms the values of ab, bc, ca.

6. With the same values of a, b, c compute $\frac{ab}{c}$.

7. With the same values of a, b, c compute $\sqrt[3]{\frac{a^2b}{c^5}}$.

8. Compute the volume of a hemispherical dome if its diameter is 150.32 feet. (Volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.)

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* C. F. GAUSS (1777–1855) was without question one of the greatest and most versatile mathematicians of all times. He was director of the observatory and professor of astronomy at Göttingen from 1807 to the end of his life.

9. If a sum of money P (the principal) is earning interest at the rate of $r\%$ a year, and if the interest is added to the principal at the end of each year, show that the amount, at the end of n years, will be

$$A = P\left(1 + \frac{r}{100}\right)^n.$$

In this case the interest is said to be compounded annually.

10. Find the amount on $157.38 for 7 years at $3\frac{1}{2}\%$ compound interest.

11. How much money must I put into the bank at 3% compound interest, so that the amount may be $500 at the end of five years?

26. The logarithms of the trigonometric functions. In solving right triangles by means of logarithms, we frequently have to find the logarithm of a trigonometric function of an angle. It would be very burdensome if we had to look up first, in the table of natural functions, the value of the function and then, from the table of mantissas, find its logarithm. In order to avoid this complication, there has been constructed an additional table which enables us to find directly the values of the logarithms of the sine, cosine, tangent, and cotangent for every angle between 0° and 90°. These quantities are denoted by the symbols log sin or L. Sin, log cos or L. Cos, etc., and are pronounced log sine, log cosine, log tangent, and log cotangent.

In the tables which accompany this book, Table II gives the values of the logarithms of the trigonometric functions directly, to five decimal places, for every minute of arc. If the angle contains fractional parts of a minute, we obtain its functions from the table by interpolation.

The arrangement of this table resembles that of the table of natural functions so closely, that it is unnecessary to describe it in detail. It should be noted, however, that in this table the characteristics of the logarithms are also given. But since the natural sines and cosines of all acute angles, and the tangents of all angles less than 45°, are proper fractions, these characteristics are negative and have been expressed in the form $9 - 10$, $8 - 10$, etc. *The continually recurring -10 has not been printed*, and should be supplied by

the computer. It is understood, once for all, that 10 is to be subtracted from all of the logarithms in the first, second, and fourth columns of the table, while the logarithms printed in the third column are provided with their correct characteristics and require no such modification.

The process of interpolation may be applied to the table of logarithms of the trigonometric functions in the same way as to the table of natural functions or to the table of logarithms of numbers.

The following examples are intended to illustrate the application of Table II.

EXAMPLE 1. Find log sin, log cos, log tan, log cot of 41° 15′ 35″.

Solution. For convenience in interpolation convert 35″ into decimal parts of a minute. Then 41° 15′ 35″ = 41° 15′.58.

We find, from the table, the following material:

41°

′	L Sin	D	L Tan	C D	L Cot	L Cos	D		P P	
—	—		—		—	—		—		15
—	—		—		—	—		—	1	1.5
15	9.81911	15	9.94299	25	0.05701	9.87613	12	45	2	3.0
16	9.81926		9.94324		0.05676	9.87601		44	3	4.5
—	—		—		—	—		—	4	6.0
—	—		—		—	—		—	5	7.5
—	—		—		—	—		—	6	9.0
—	—		—		—	—		—	7	10.5
—	—		—		—	—		—	8	12.0
—	—		—		—	—		—	9	13.5
	L Cos	D	L Cot	C D	L Tan	L Sin	D	′		

48°

We conclude:

log sin 41° 15′.58 = 9.81911 + .58 of 15 units of the 5th decimal place.
log tan 41° 15′.58 = 9.94299 + .58 of 25 units of the 5th decimal place.
log cot 41° 15′.58 = 0.05701 − .58 of 25 units of the 5th decimal place.
log cos 41° 15′.58 = 9.87613 − .58 of 12 units of the 5th decimal place.

We may use the marginal tables of proportional parts to complete the interpolation. Thus, the table headed 15, shows that .5 of 15 is 7.5 and

.08 of 15 is 1.2, and consequently .58 of 15 is 8.7 or 9 units of the fifth decimal place. Therefore

$$\log \sin 41° \, 15'.58 = 9.81920 - 10.$$

In the same way we find

$$\log \tan 41° \, 15'.58 = 9.94314 - 10, \quad \log \cot 41° \, 15'.58 = 0.05686,$$
$$\log \cos 41° \, 15'.58 = 9.87606 - 10.$$

EXAMPLE 2. Find the logarithms of the functions of 48° 44'.42.

Solution. This angle is the complement of that of Example 1. Hence each of its functions is equal to the corresponding cofunction of 41° 15'.58, and the values obtained are the same as in Example 1 with the name of each function changed to the corresponding cofunction.

Just as in the table of natural functions, these values, for angles greater than 45°, may be obtained directly from the table by reading the degrees of the angle at the *foot* of the page, the minutes in the right-hand column, and the name of the function at the foot of each of the four columns. We find, in this way,

$$\log \sin 48° \, 44'.42 = 9.87606 - 10, \quad \log \cot 48° \, 44'.42 = 9.94314 - 10,$$
$$\log \tan 48° \, 44'.42 = 0.05686, \quad \log \cos 48° \, 44'.42 = 9.81919 - 10.$$

EXAMPLE 3. Given $\log \tan A = 0.53219$. Find A.

Solution. The given logarithm does not appear anywhere in the column at the foot of which is printed the name L Tan. But we do find in this column

$$\begin{aligned} \log \tan 73° \, 38' &= 0.53212, \\ \log \tan 73° \, 39' &= 0.53259. \\ \hline \text{Tabular difference for } 1' &= 0.00047. \end{aligned}$$

The given value of $\log \tan A$ is $\frac{7}{47}$, or $\frac{15}{100}$, of the way from the first toward the second of these tabular logarithms. Therefore

$$A = 73° \, 38'.15$$

27. The accuracy of five-place tables. As we have said repeatedly, the number of decimal places used in stating the result of a measurement is to be regarded as an indication of its degree of precision. We shall ordinarily wish to make all calculations, based upon such measurements, with a sufficient number of decimal places to avoid introducing inaccuracies which might have an appreciable influence upon the results, *i.e.* an influence comparable with that produced by the unavoidable errors of observation.

Five-place tables are quite accurate enough to satisfy this condition in almost all problems of engineering and natural

science. In fact, in most problems of this kind, the distances are not measured so accurately as to exclude an error of one twentieth of one per cent of their value, and the angles are read to the nearest minute only. But five-place tables are far more accurate than this. In fact, a distance expressed by a five-place number presupposes an accuracy of at least $\frac{1}{200}\%$, and an angle may usually be determined from five-place logarithms of its functions with an error of not more than 2 or 3 seconds of arc.

We must not forget, however, that the degree of accuracy of a table is not the same in all parts of the table, and that we must use our judgment in the selection of the formulæ which we wish to use in solving a problem.

28. The trigonometric functions of angles near 0° or 90°. Thus, for instance, if we wish to determine an angle for which $\log \cos A = 9.99998 - 10$, our table cannot furnish an accurate result. We find, by referring to the table, that A may have any value between $0^\circ\ 29'$ and $0^\circ\ 36'$.

A small angle cannot be determined, with any degree of accuracy, from the value of its cosine.

In the same way, we see that *an angle very close to* 90° *cannot be determined accurately from the value of its sine.*

In most cases we shall be able to modify the formula, which we are using, in such a way as to avoid this difficulty. If, for instance, the angle A (known to be very small) is to be determined from the value of its cosine, we shall seek some other formula as a solution of the same problem by means of which the angle A can be determined from the value of its sine or tangent. The problem then reduces to that of finding a small angle when its sine or tangent is given. If we again refer to our table, we find that this problem also gives rise to a difficulty. The method of interpolation, which we ordinarily use, becomes both cumbersome and inexact in the case of such small angles, because the tabular differences are very large and change very rapidly from one place in the table to another.

In order to meet this difficulty, we have provided a separate table (Table III), giving the values of the logarithmic functions for every second of arc from 0° 0′ to 0° 3′ and from 89° 57′ to 90°, and for every ten seconds from 0° to 2° and from 88° to 90°.

Another method of meeting this difficulty, preferable in some respects, will be explained in the second part of this book (Art. 85); it involves the auxiliaries S and T (Table IV).

EXERCISE XIV

1. Find log sin 15° 6′, log cos 35° 13′, log tan 57° 28′, log cot 76° 44′.

2. Find log sin 18° 23′.35, log tan 41° 46′.27, log cos 64° 17′ 43″, log cot 25° 12′ 38″.

3. Find the angles for which

$$\log \sin A = 9.42553 - 10, \quad \log \cos B = 9.60618 - 10$$
$$\log \cot C = 9.68497 - 10, \quad \log \tan D = 0.14193.$$

4. Find the angles for which

$$\log \cot A = 0.11157, \quad \log \tan B = 9.75465 - 10,$$
$$\log \cos C = 9.68334 - 10, \quad \log \sin D = 9.56652 - 10.$$

5. Find log sin 0° 2′ 15″, log tan 1° 10′ 22″.

6. Find the angles for which

$$\log \sin A = 5.83170 - 10, \quad \log \tan B = 8.32313 - 10.$$

7. Find, by logarithms, the angle A, if $\tan A = a/b$ and $a = 1.2291$, $b = 14.950$.

29. The logarithmic or Gunter scale. No graphical process is more familiar than the addition and subtraction of line-segments, and this process may evidently be used as a substitute for addition and subtraction of numbers. Since addition of logarithms corresponds to multiplication of numbers, we may find the logarithm of a product graphically by adding line-segments, whose lengths are equal to the logarithms of the factors.

In order to do this, we must have some means for actually finding a line-segment whose length shall be equal to the logarithm of a given number.

Let us take a line-segment of convenient length, say 10 centimeters, as unit of length. In terms of this unit, the whole distance (10 centimeters = 100 millimeters) represents log 10, since the logarithm of 10 is equal to unity. If we count all distances from the left-hand end of the line, we may label the right-hand end 10 to indicate that this distance represents log 10. The left-hand end will then be labeled 1, because $\log 1 = 0$.

From the table of logarithms we have, to two decimal places,

$$\begin{aligned} &\log 1 = 0.00, \quad \log 2 = 0.30, \quad \log 3 = 0.48, \quad \log 4 = 0.60,\\ (1)\quad &\log 5 = 0.70, \quad \log 6 = 0.78, \quad \log 7 = 0.85, \quad \log 8 = 0.90,\\ &\log 9 = 0.95, \quad \log 10 = 1.00. \end{aligned}$$

We mark the points on our line-segment whose distances from the left-hand end, measured in terms of the whole line as unit, are in order equal to log 2, log 3, log 4, ... log 9, and label them 2, 3, 4, ... 9, respectively. If the whole line-segment is 10 centimeters long, these points will, on account of (1), be at distances 30, 48, 60, 70, 78, 85, 90, 95 millimeters, respectively, from the left-hand end of the line-segment (cf. Fig. 14).

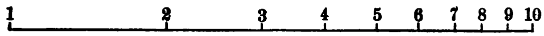

FIG. 14

A scale constructed in this way is called a **logarithmic scale**, and its usefulness for purposes of calculation was first pointed out by EDMUND GUNTER* in 1620. It enables us to find a line-segment equal in length to the logarithm of any number between 1 and 10. It is easy to see how, by means of such a scale and a pair of dividers, multiplication and division may be reduced to the simple graphical processes of adding and subtracting line-segments.

30. The slide rule. Some years before 1630, WILLIAM OUGHTRED † noticed that the use of the dividers might be avoided by constructing two equal logarithmic scales (Scales

* Professor of astronomy in Gresham College, London (1581–1626).

† OUGHTRED (1575–1660) was a fellow of King's College, Cambridge.

A and *B* of Fig. 15), capable of sliding by each other, as indicated in the figure.*

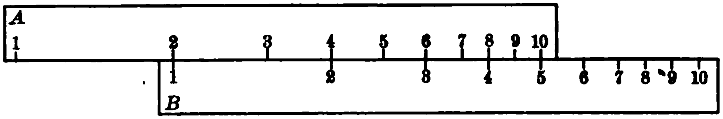

FIG. 15

The use of this simple bit of apparatus for the purpose of multiplication and division will be apparent from the following examples :

To multiply 2 by 3. Place scale *B* in such a way that its left-hand index (*i.e.* the division marked 1) falls directly under the division marked 2 on scale *A*. Directly above the division marked 3 on scale *B*, we shall find, on scale *A*, the product which (of course) is 6. To justify this process it suffices to note that it is equivalent to adding the logarithm of 3 to that of 2.

Fig. 15 shows scales *A* and *B* in the proper position for the purposes of this example.

To divide 6 by 3. Under the division 6 of scale *A*, place division 3 of scale *B*. Over the division 1 of scale *B* we shall find the quotient ($\frac{6}{3} = 2$) on scale *A* (cf. Fig. 15).

The instrument actually in use, the MANNHEIM slide rule, is a slight amplification of the one just described (cf. Fig. 16). It has four scales, usually denoted by *A*, *B*, *C*, *D*, respectively, the scales *A* and *D* being on the rule, and *B* and *C* on the slide.

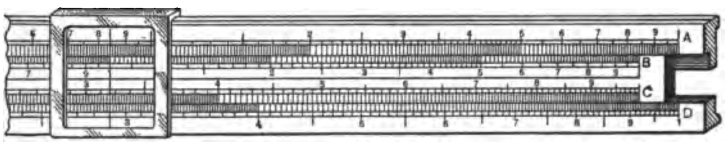

FIG. 16

The scale *A* is composed of two logarithmic scales such as that of Fig. 14, so that its right-hand end might be labeled 100, since log 100 = 2. On most slide rules, however, the first principal division on scale *A* after 9 is not labeled 10,

* Oughtred's instruments were described in publications of WILLIAM FOSTER, one of his pupils, in 1632 and 1633.

as in Fig. 14, but 1, the next one is not labeled 20, but 2, and so on to the last one, which is again labeled 1 instead of 100 or 10. Thus, the two halves of scale *A* are exact copies of one another. This is done for precisely the same reason that the mantissas only are printed in our tables of logarithms. The slide rule also makes use of the mantissas only. The characteristics, or what amounts to the same thing, the position of the decimal point in the result, must be obtained by inspection or by special rules.

Scale *B* is on the upper edge of the slide, in direct contact with scale *A* on the rule, and is an exact copy of scale *A*. These two scales together may be used for multiplication and division as explained above.

Scale *D* is on the lower part of the rule. It is a single logarithmic scale, from 1 to 10, of the same length as the combined two scales of A. The logarithm of any number is therefore represented, on scale *D*, by a distance twice as great as that which represents the logarithm of the same number on scale *A*. It follows from this that the number which is found on scale *A*, vertically above any number of scale *D*, is the square of the latter. Any number on scale *D*, on the other hand, is the square root of the number vertically above it on scale *A*.

Scale *C* is on the lower edge of the slide, in direct contact with slide *D* on the rule. It is an exact copy of scale *D*. These two scales together may be used for multiplication and division, according to the same rules which hold for scales *A* and *B*.

Besides these four scales, the slide rule is supplied with a *runner* (cf. Fig. 16), which is useful in performing compound operations, and also in comparing two scales (such as *A* and *D*), which are not in direct contact with each other. The runner was made a permanent feature of the slide rule by MANNHEIM in 1851.*

* AMÉDÉE MANNHEIM (1831–1906), a distinguished geometer of recent times. The runner had however been used occasionally, long before Mannheim, by a number of English mathematicians.

It often happens, in manipulating the slide rule, that the result is to be sought opposite a number of the slide which falls outside of the scale on the rule. In such cases, we may *shift* the slide, bringing the right-hand index to the place which the left-hand index occupied previously, and read off the result as before. For, such a shift has no influence on the mantissa, since it merely amounts to dividing the result by 10. On the Mannheim rule, this shifting of the slide may be avoided by working with scales *A* and *B* rather than with *C* and *D*. Scales *C* and *D*, however, have the advantage of greater accuracy.

If the slide be withdrawn entirely, it will be found to have three other scales on its reverse side, two of which are labeled *S* and *T*. These are scales of logarithmic sines and tangents, respectively, and may be used for calculating such products as

$$c \sin A, \quad c \tan A.$$

The middle scale on the reverse side is used for finding the value of the logarithm of a number, and is important if we wish to compute a power of a number with a complicated fractional exponent.

For more complete information concerning the slide rule, we must refer to the manuals which are usually presented to the purchaser of such an instrument.* Cheap slide rules, especially constructed for the beginner, may now be obtained of all dealers under the name Student's or College Slide Rule. Engineers and computers use the slide rule so extensively that the student will find it advisable to make himself familiar with the instrument by actual use.

The Mannheim slide rule, which we have described, admits of three-figure accuracy. In some (exceptional) cases, results correct to four decimal places may be obtained by its use. The THACHER and FULLER slide rules, more complicated instruments, but constructed on essentially the same

* See also RAYMOND's Plane Surveying.

principles, admit of far greater accuracy. The EICHHORN Trigonometric Slide Rule was invented for the purpose of solving triangles, and is especially adapted for this work. But, of course, it has not the wide range of usefulness of the ordinary slide rule.

CHAPTER VI

APPLICATION OF LOGARITHMS TO THE SOLUTION OF RIGHT TRIANGLES

31. The general method. We have shown, in Chapter III, how to solve right triangles by means of the natural functions, and we have become acquainted with the theory and use of logarithms in Chapter V. To solve a right triangle by logarithms, it suffices to combine the results of these two chapters. We use the same formulæ as in Chapter III, but perform the multiplications and divisions by means of logarithms, using the table of logarithmic sines, cosines, etc., in place of the table of natural functions.

In order to illustrate the various practical questions which arise in such a calculation, we shall give a rather extended discussion of the following example:

EXAMPLE. The legs of a right triangle were found to be $a = 527.38$ feet and $b = 621.24$ feet. Calculate the hypotenuse and the acute angles A and B.

32. The preliminary graphic solution.

We first make a drawing, approximately to scale, making

$$a = 5.3 \text{ centimeters},$$
$$b = 6.2 \text{ centimeters}.$$

We find by measurement

$$c = 8.1 \text{ centimeters},$$
$$A = 40°.5, \quad B = 49°.5.$$

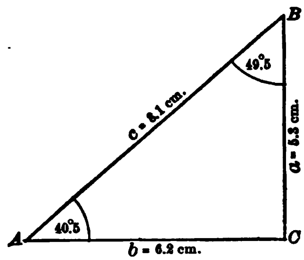

FIG. 17

This figure and these measurements serve two purposes. In the first place, the figure helps us pick out the formulæ which we shall need for the trigonometric solution of our triangle. In the second place, comparison of the approximate values of the unknown quantities obtained graphically,

with the final results as obtained by calculation, constitutes a valuable check. If the results obtained by the two methods should differ by more than can be accounted for by the inaccuracies of the graphic solution, we must look for a mistake in our calculation.

33. The gross errors. Mistakes, which are large enough to be detected by means of a graphic check, are known as *gross errors* and are usually due to one of the following causes:

1. The use of a wrong formula.
2. A misplaced decimal point or, what amounts to the same thing, an erroneous characteristic.
3. The use of a number taken from a wrong column in the tables, resulting, for instance, in erroneously using the log cos of an angle in place of the log sin.
4. Addition of two logarithms when subtraction is required or *vice versa*.
5. Purely arithmetical errors of addition and subtraction.

The errors of the first four classes can be avoided by the exercise of a sufficient amount of care. The student should not attempt to gain speed in calculation until he has first learned to be accurate. The errors of the last class are quite unavoidable, but they will usually be detected almost as soon as made if the ordinary arithmetical checks for addition and subtraction be applied every time that one of these operations is used.

34. Selection of formulæ and checks. After completing the approximate graphical solution of a triangle, we pick out the formulæ which we wish to use in the computation.

In our example, these are the following:

$$\tan A = \frac{a}{b},\quad c = \frac{a}{\sin A} = \frac{b}{\cos A},\quad B = 90^\circ - A, \tag{1}$$

or, in logarithmic form,

$$\begin{aligned} \log \tan A &= \log a - \log b, \qquad B = 90^\circ - A, \\ \log c &= \log a - \log \sin A = \log b - \log \cos A. \end{aligned} \tag{2}$$

We have two formulæ for c, and in most cases it makes little difference which one we decide to use. If we were to use both, the agreement

of the two results would constitute a partial check on the accuracy of our work. It would not be a total check however; a mistake in the logarithm of a or b could not be detected by means of it.* It is hardly worth while therefore to use both formulæ for c. That one is to be preferred which has the greater denominator, as the result obtained from it is likely to be the more accurate.

For a complete check, we may make use of the equation

$$a^2 + b^2 = c^2,$$

which, however, we prefer to write in the form

$$a = \sqrt{c^2 - b^2} = \sqrt{(c + b)\ (c - b)}, \tag{3}$$

which is more convenient for logarithmic computation.

It should not be necessary to write the formulæ in the logarithmic form (2). Form (1) is shorter and more directly connected with the geometry of the problem. Moreover, it contains all of the information that is necessary for the solution of the problem, for anybody who has studied logarithms.

35. The framework or skeleton form. Having selected the formulæ, we proceed to plan the details of the computation by providing a definite, properly marked place for every number which will be needed in the course of the work. Moreover, we shall plan these details in such a way that those numbers which are to be combined by addition or subtraction will have their places in the same vertical column next to each other.

In our particular example we may adopt the following framework:

Given	$a =$	(1)
	$b =$	(2)
	$\log a =$	(3)
	$\log b =$	(4)
	$\log \tan A =$	(3) − (4) = (5)
Results	$A =$	(6)
	$B =$	(16)
	$c =$	(9)

	$\log b =$	(4)
	$\log \cos A =$	(7)
	$\log c =$	(4) − (7) = (8)
	$c =$	(9)
	$b =$	(2)
	$c - b =$	(9) − (2) = (10)
	$c + b =$	(9) + (2) = (11)
	$\log (c - b) =$	(12)
	$\log (c + b) =$	(13)
	$\log (c^2 - b^2) =$	(12) + (13) = (14)
Check	$\log \sqrt{(c^2 - b^2)} =$	$\frac{1}{2}$ (14) = (15)
	$\log a =$	(3)

* Since such a mistake could be interpreted as leading to the correct solution of a triangle different from the given one, namely, that one whose sides a' and b' have as logarithms the values which were, by mistake, assigned to a and b.

The numbers in parenthesis merely indicate the order in which this skeleton form may be filled in, and how some of the results are obtained. The student should use these numbers only to aid him in understanding the construction of the framework and the plan of the computation. They should not be used in writing out the actual calculation.

36. The computation. We are now prepared to carry out the computation. This should be done on paper ruled into squares of such a size that each figure may conveniently occupy one square. We obtain the following results:

$$\text{Given}\begin{cases} a = 527.38 \\ b = 621.24 \end{cases}$$

$$\begin{aligned} \log a &= 2.72212 \\ \log b &= 2.79326 \\ \log \tan A &= 9.92886 - 10 \end{aligned}$$

$$\text{Results}\begin{cases} A = 40^\circ\, 19'.69 \\ B = 49^\circ\, 40'.31 \\ c = 814.92 \end{cases}$$

$$\begin{aligned} \log b &= 2.79326 \\ \log \cos A &= 9.88215 - 10 \\ \log c &= 2.91111 \\ c &= 814.92 \\ b &= 621.24 \\ c - b &= 193.68 \\ c + b &= 1436.16 \\ \log (c - b) &= 2.28709 \\ \log (c + b) &= 3.15720 \\ \log (c^2 - b^2) &= 5.44429 \end{aligned}$$

$$\left.\begin{aligned} \log \sqrt{c^2 - b^2} &= 2.72215 \\ \log a &= 2.72212 \end{aligned}\right\}\text{Check.}$$

Remark. We observe that the check is not absolute. The agreement is as close, however, as we should expect. The inevitable inaccuracies, arising from the neglected higher decimal places, often manifest themselves by discrepancies amounting to several units of the fifth decimal place. Consequently, we may declare the check to be satisfactory.*

37. Revision of the computation when the check is unsatisfactory. If, in the solution of such an example, the results fail to check satisfactorily, the magnitude of the discrepancy will help us to locate the error. If the discrepancy is very great, the error must be one of the *gross* kind which we have discussed in Art. 33. In case of a comparatively small discrepancy, our error is probably due to one of the following causes:

1. Purely arithmetical errors of addition and subtraction in the last few decimal places.

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* If a and b differ considerably, use as check $a = \sqrt{(c - b)(c + b)}$ or $b = \sqrt{(c - a)(c + a)}$, according as $b < a$ or $b > a$.

2. Inexact interpolation, which would ordinarily affect only the last decimal place.

3. Addition of the correction obtained by interpolation when it should be subtracted, or *vice versa*.

This last mistake may be avoided by carefully inspecting the table *after* the interpolation has been completed, so as to make sure that the quantity calculated actually lies between the two numbers of the table between which it *should* fall.

EXERCISE XV

In each of the following examples (1–10), two parts of a right triangle are given in the usual notation. Find the other parts:

1. $c = 627,\quad A = 23° 30'$.
2. $c = 934,\quad B = 76° 25'$.
3. $a = 637,\quad A = 4° 35'$.
4. $b = 48.532,\quad B = 36° 44'.00$.
5. $a = 38.313,\quad b = 19.522$.
6. $a = 13.690,\quad b = 16.926$.
7. $a = 67.291,\quad c = 110.970$.
8. $b = 618.42,\quad c = 1843.70$.
9. $a = 965.24,\quad A = 75° 15'.2$.
10. $a = 7.3298,\quad b = 6.1032$.

An isosceles triangle may be divided into two equal right triangles by dropping a perpendicular from the vertex to the base. Using the notations of Fig. 18, find the missing sides and angles of the following isosceles triangles. (Exs. 11–13.)

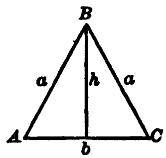


FIG. 18

11. $b = 2.1452,\quad B = 121° 14'.60$.
12. $A = 52° 10'.2,\quad a = 600.20$.
13. $h = 7.447,\quad A = 76° 14'.00$.
14. Prove that the area S of a right triangle is
$$S = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} ac \cos A.$$
15. Prove that the area S of a right triangle is
$$S = \tfrac{1}{2} c^2 \sin A \cos A.$$

16–25. Find the area of each of the right triangles in Exs. 1–10.

26–30. Find formulæ for the area of the isosceles triangle of Fig. 18, in terms of b and h; a and b; a and h; a and B; a and A.

31–33. Apply the results of Exs. 26–30 to find the areas of the isosceles triangles of Exs. 11–13.

34. Show that the perimeter p of a regular polygon of n sides inscribed in a circle of radius R is
$$p = 2\,nR \sin \frac{180°}{n},$$

that the radius r of the inscribed circle is

$$r = R \cos \frac{180^\circ}{n},$$

and that the area S of the polygon is

$$S = nR^2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}.$$

35. Find the radius of the inscribed circle, the perimeter and the area of a regular pentagon, if the radius of the circumscribed circle is 12 feet.

36. Find the perimeter, the length of one side, the radii of the inscribed and circumscribed circles of a regular octagon whose area is 24 square feet.

37. Since the polygon of Ex. 33 approaches the circle of radius R as limit when n grows beyond all bound, what limits do $n \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}$ and $2n \sin \frac{180^\circ}{n}$, respectively, approach?

38. Applications to simple problems of surveying, navigation, and geography. The connection between surveying and trigonometry is so obvious as to require no further explanation. Moreover, we have already discussed this relation in Chapter 1.

Many of the following examples are concerned with simple problems of surveying, and most of the technical terms which occur in them are self-explanatory. Nevertheless, we shall give a brief discussion of these terms, so as to make the applications seem more concrete and vivid. The student who wishes to know more about the subject should consult a treatise on surveying.*

A **plumb line** is a cord to one end of which is attached a weight. If such a plumb line be suspended, by fastening the other end of the cord to a fixed support, it will oscillate to and fro, and finally come to rest in its position of equilibrium, which is called the **vertical line** of the place of observation.

Since the earth is approximately spherical in shape and the plumb line points toward the earth's center, the vertical lines of different places are not parallel. But the angle between

* For instance, RAYMOND's Plane Surveying.

the vertical lines of two stations which are not very far apart (say ten miles), is so small that for most purposes these lines may be regarded as parallel. When a tract of land (to be surveyed) is comparatively small, it is therefore legitimate to neglect the effect of the earth's curvature, and the problem becomes one of **plane surveying.** The more difficult problems connected with a *geodetic survey*, in which the earth's spherical form is taken into account, require knowledge of the methods of *spherical trigonometry.*

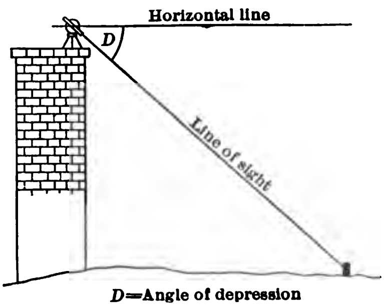
We are concerned with plane surveying only, so that we shall regard the vertical lines of all places which occur in such a survey as parallel.

A **vertical plane** is one which contains a vertical line.

A **horizontal line** or **plane** is one which is perpendicular to a vertical line.

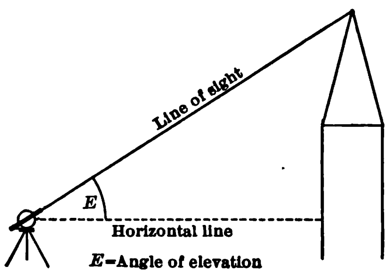
An inclined line or **plane** is one which is neither vertical nor horizontal.

An angle is said to be a **horizontal or vertical angle**, according as the plane of its sides is a horizontal or vertical plane. Both horizontal and vertical angles may be measured by means of the transit (Art. 2). Inclined angles may be measured by means of an instrument known as a *sextant.*



D=Angle of depression

Fig. 19



E=Angle of elevation

Fig. 20

The angle which the line of sight from the observer to an object makes with a horizontal line, in the same vertical plane, is called the **angle of elevation** or the **angle of depres-**

sion, according as the object is above or below the horizontal plane of the observer (cf. Figs. 19 and 20).

The **angle subtended by a line** is that which is obtained by joining the extremities of the line to the eye of the observer.

The direction or **bearing** of any horizontal line is usually described by means of the angle which it makes with the north-south line or **meridian**, the latter being located approximately with the help of a surveyor's **compass**. Surveyors always measure the bearing of a line as an acute angle from the north or south end of the meridian toward the east or west point, as the case may be. Thus, in Fig. 21, the bearing of OA is N 45° E, that of OB is S 10° W, that of OC is N 30° W.

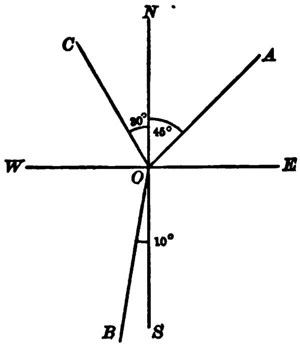


FIG. 21

Surveyors usually measure distances by means of a Gunter's chain, which is 4 rods or 66 feet long, and is divided into 100 links. For this reason, the operation of measuring the length of a line in the field is frequently called **chaining**.

In order to measure the difference of level between two places, A and B (cf. Fig. 22), the observer at O first makes his telescope point in a horizontal direction by means of a spirit level attached to the telescope. An assistant holds a graduated rod, R, in a vertical position at A, and the observer at O reads, by means of the telescope, the division on the graduated rod where it is struck by the horizontal line of sight.

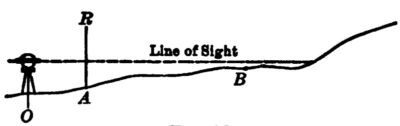


FIG. 22

He repeats this operation with the rod at B. The difference between the two readings gives the difference of level between A and B. Of course, if the difference of level between A and B exceeds the length of the rod, intermediate stations must be introduced. This operation is known technically as **leveling**.

The **navigator** does not always express bearings in the same language as the surveyor. He divides the circumference into 32 equal parts, called points of the compass. Thus one point of the compass is an angle of $11\frac{1}{4}$ degrees. The division points are named, as indicated in Fig. 23, with obvious reference to the four cardinal points of the compass, — north, south, east, and west.

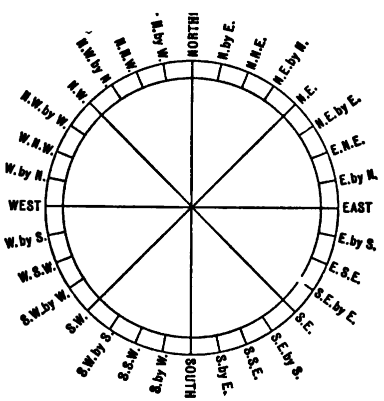


FIG. 23. — The points of the Mariner's Compass

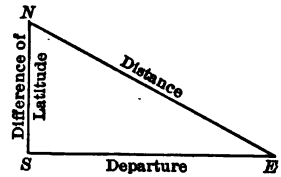


FIG. 24

The navigator also makes use of the terms **departure** (to denote the east and west component of a course), and **difference in latitude** (to denote the north and south component). These terms are illustrated in Fig. 24.

EXERCISE XVI

In solving the following problems, the student should exercise his judgment in regard to the number of decimal places to be used in the calculation. (See Chap. I, Arts. 1, 2; Chap. IV, Arts. 14, 15.) Many of these problems may be solved by means of three place tables (Tables IX, X, and XI of our collection), or by means of the slide rule. (See Chap. V, Arts. 29, 30.) In all of the examples the slide rule may be used as a check.

1. At a point 180.00 feet away from the base of a tower and in the same horizontal plane with it, the angle of elevation of the top was found to be 65° 40′.5. Find the height of the tower.

2. From the top of a cliff 120 feet above the level of a lake, the angle of depression of a boat was found to be 27° 40′. What is the air line distance from the top of the cliff to the boat?

3. In order to measure the width of a river, a base line AC is measured along one bank 215.6 feet long. By means of a transit, a point B is located on the opposite bank such that ACB is a right angle. The angle BAC is found to be $55° \ 16'.2$. What is the width BC of the river?

4. From the top of a mountain 2653 feet above the floor of the valley, the angles of depression of two farmhouses in the level valley beneath, both of which were due east of the observer, were found to be 25° and 56°. What is the horizontal distance between the two houses?

5. From the top of a hill, the angles of depression of two consecutive milestones on a straight level road, running due south from the observer, were found to be 22° 31′ and 48° 15′. How high is the hill?

HINT. Treat this problem as one involving two unknowns: 1st, the height of the hill; 2d, the horizontal distance from one of the milestones to the foot of the perpendicular dropped from the top of the hill to the horizontal plane of the road.

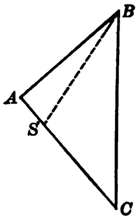


FIG. 25

6. Three lighthouses A, B, C, are situated as in Fig. 25, the triangle ABC being right-angled at A. At the moment when a ship S is crossing the line AC, the angle ASB is found to be 75°. If the distance between the lighthouses A and B is 12 miles, what are the distances from S to A and B?

7. A light on a certain steamer is known to be 35 feet above the water. An observer on the shore, whose instrument is 5 feet above the water, finds the angle of elevation of this light to be 5°. What is the distance from the observer to the steamer?

8. What angle does a mountain slope make with a horizontal plane, if it rises 200 feet in a horizontal distance of one tenth of a mile?

9. The cable of a captive balloon is 835 feet long. Assuming the cable to be straight, how high is the balloon when all of the cable is out if, owing to the wind, the cable makes an angle of 25° with a vertical line?

10. A ship is sailing due west at the rate of 8.9 miles per hour. A lighthouse is observed due south at 10 P.M. The bearing of the same lighthouse at 11:55 P.M. was S. 34° E. Find the distance from the lighthouse to the ship at the time of the second observation.

HINT. In Fig. 26, S and S' represent the two positions of the ship and L represents the lighthouse. Angle $SS'L = 90° - 34°$.

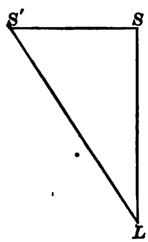


FIG. 26

11. Find the area of the tract of land corresponding to the following description. From A the boundary line runs N. 24° E. 20 chains to B, thence N. 85° W. 35.67 chains to C, and thence back to A.

12. The shadow of a chimney, 50 feet high, is 60 feet long. What is the altitude (or angle of elevation) of the sun at that instant?

13. The last row of seats in a circular tent is 20 feet away from the central pole, which is 18 feet high, and which is to be fastened by ropes from its top to stakes driven in the ground. How long must these ropes be in order that they may be 6 feet above the ground over the last row of seats, and at what distance from the center must the stakes be driven?

14. How long must a ladder be to reach a window 45 feet high, directly above a porch 15 feet high, if the porch projects 10 feet from the building?

15. A building 125 feet high, with a flat roof, faces north on a boulevard. The distance from this building to the one directly opposite is 180 feet. How far back from the edge of the roof should a chimney 6 feet high be placed, so as to be invisible from any point on the boulevard due north of the chimney?

16. A cylindrical pipe 36 inches in diameter is to be joined to a second cylindrical pipe 18 inches in diameter. The axes of the two cylinders are pieces of the same horizontal line and their ends are 6 feet apart. The joining piece is to be in the form of a frustum of a cone. Draw a sectional view of the joining piece and compute the length of the slanting side and the angles.

17. We wish to construct a house with a gable roof. If the house is 25 feet wide, if the height under the eaves is 27 feet and the height to the ridge pole 35 feet, how long must the rafters be so that their ends may be at a horizontal distance of 2 feet from the side of the house?

18. The angle of elevation of the center of a spherical balloon 20 feet in diameter was found to be 65°. The angle which it subtended at the same time was 2° 30′. What is the height of the balloon above the horizontal plane of the observer?

19. Two stations, A and B, are to be connected by a railroad. Both stations are in the same horizontal plane, and B is 35 miles northeast of A. The two stations are separated by a lake, which terminates at a point C, 12 miles north and 29 miles east of A. Find the lengths and bearings of the two portions of the road from A to C and from C to B.

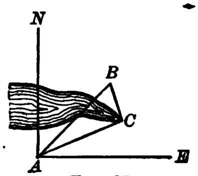


FIG. 27

20. A lecture room, 50 feet long and 18 feet high, is to be supplied with a sloping floor. The front part of the floor, for the first ten feet, is

to be horizontal so as to admit of the placing of lecture tables and apparatus. The highest part of the sloping floor, at the back of the room, is to be 8 feet from the ceiling. What length of sloping timbers is required for this construction? Each of these timbers is to be supported at both ends and by six intermediate uprights placed at equal horizontal distances from each other and from the end supports. How far should each of these eight supports project above the horizontal part of the floor?

21. In order to determine the height of a mountain above a level plane, we may measure a horizontal base line of length b in the same vertical plane with the summit of the mountain and observe the angles of elevation, A and B, of the summit from the two ends of the base line. Find a formula for the vertical height h of the mountain above the level of the plane.

22. Apply the formula of Ex. 21 to the case $b = 100$ feet, $A = 30°$, $B = 35°$.

23. A flagstaff, known to be h feet in length, stands on top of a cliff. An observer, in the same horizontal plane with the base of the cliff, finds the angles of elevation of the top and bottom of the flagstaff to be A and B respectively. Find a formula for the height of the cliff.

24. Apply the formula of Ex. 23 to the case $h = 25$, $A = 40° 25'$, $B = 37° 10'$.

25. The angle of elevation of the top of a spire from the third floor of a building was $35° 10'$. The angle of elevation from a point directly above, on the fifth floor of the same building, was $25° 33'$. What is the height of the spire and its horizontal distance from the place of observation, if the distance between consecutive floors is 12 feet and the first floor rests on a basement 5 feet above the level of the street?

26. Prove the following statement. If R is the radius of the earth regarded as a sphere, the radius of the parallel of latitude which passes through a place P of latitude L, is

$$r = R \cos L.$$

HINT. Use Fig. 28, where O denotes the earth's center, N and S the north and south poles, $OE = R$ the radius of the equator, $\angle EOP = L$ the latitude of the place P, and $r = MP$ the radius of the parallel of latitude which passes through P.

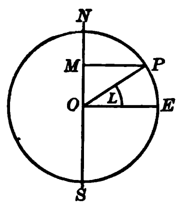


FIG. 28

27. Let d be the length, in miles, of a degree of longitude at the equator. Show that the length of a degree of longitude, at latitude L, will be $d \cos L$.

28. Show that the radius r (in feet) of the horizon of an observer, h feet above the earth's surface, is given by the formula

$$r = \frac{R}{R+h}\sqrt{h\,(2\,R+h)},$$

if R denotes the earth's radius expressed in feet.

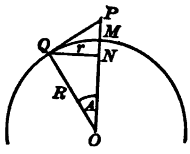


FIG. 29

HINT. Use Fig. 29, where $OQ = OM = R$, $MP = h$, $QN = r$, using the angle $NOQ = A$ as auxiliary.

29. A micrometer screw is to be cut from a cylindrical steel rod, 5 millimeters in diameter, in such a way that one complete revolution of the screw will move the wire W, attached to the movable frame F (Fig. 30), through a distance of one millimeter. What angle will the thread of the screw make with a plane perpendicular to the axis of the screw?

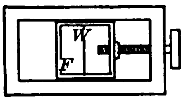


FIG. 30

30. A street railway track is d feet from the curbstone. In passing a corner (Fig. 31), where the street is deflected through an angle of $K°$, it is desired to have the rail pass at a distance of d' feet from the corner. Show that the radius of the circular curve ACB must be

$$r = \frac{d - d' \cos\frac{K}{2}}{1 - \cos\frac{K}{2}}.$$

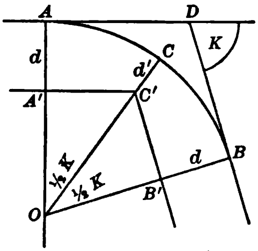


FIG. 31

31. Solve the problem of Ex. 30 numerically for the cases $d = 10$ feet, $d' = 4$ feet, $K = 90°$; and $d = 9$ feet, $d' = 3.5$ feet, $K = 60°$.

32. In order to find the horizontal distance between the points A and F which are at different levels and situated on opposite sides of a rolling valley (see Fig. 32), the distances $AB = d_1$, $BC = d_2$, ... $EF = d_5$ are measured along the ground by chaining. A transit is placed at O, and $A'B'C' \cdots F'$ represents the line of sight of the instrument. This line of sight is made horizontal by means of a spirit-level attached to the telescope. The vertical distances

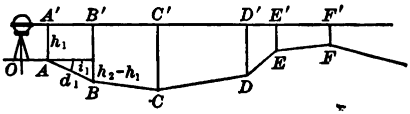


FIG. 32

$$AA' = h_1,\ BB' = h_2,\ CC' = h_3,\ DD' = h_4,\ EE' = h_5,\ FF' = h_6$$

are measured by a rod. (Cf. Fig. 22 for method of using rod.)

Show that the distance

$$A'F' = d_1 \cos i_1 + d_2 \cos i_2 + d_3 \cos i_3 + d_4 \cos i_4 + d_5 \cos i_5,$$

where i_1, i_2, i_3, i_4, i_5 are the angles of inclination of AB, BC, CD, DE, EF, respectively. The angle i_1 is determined by the equation

$$\sin i_1 = \frac{h_2 - h_1}{d_1}.$$

The angles $i_2 \cdots i_5$ are determined in similar fashion.

33. In constructing a telegraph line across a hill $ABC, \cdots I$ (Fig. 33), posts were set at $A, B, C, \cdots I$, these points being determined by *level chaining** in such a way that the horizontal distance between any two of them $A'B' = B'C' = C'D' = \cdots = H'I' = d$ feet. By leveling, the elevations of the points A, B, C, etc., were found to be

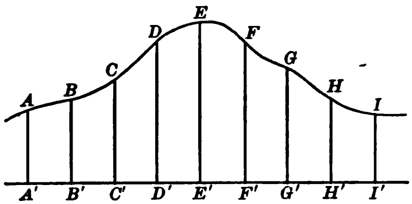


FIG. 33

$$AA' = h_1,\ BB' = h_2,$$
$$CC' = h_3, \cdots II' = h_9.$$

Find a method for computing the amount of wire required between A and I, assuming that the telegraph poles are vertical and of the same height, and making no allowance for sag.

39. Right triangles of unfavorable dimensions. If the hypotenuse and one side (say b and c) are given, we have the equation

$$(1) \qquad \cos A = \frac{b}{c}$$

to determine the angle A. But, if b differs very little from c, the value of $\cos A$ will be very close to unity and, as we observed in Art. 28, it will be impossible to determine A with any degree of accuracy from this equation.

A surveyor will usually (not always) be in a position to avoid this difficulty. For he has a certain amount of liberty in the choice of his triangles. But, in many problems of astronomy and mathematical geography, no such choice is possible, so that it becomes a matter of practical importance to find a formula for determining the angle A, which shall not be liable to the same objection as (1).

* In level chaining, the surveyor's chain or tape is held in a horizontal position, so as to measure the *horizontal* distance between two points and not the distance along the slope.

Such a formula may be obtained as follows. In Fig. 34, draw AD, the bisector of the angle A, and also BE perpendicular to AD. Then

$$\tfrac{1}{2} A = \angle CAD = \angle CBE$$

(both angles being complementary to $\angle AEB$), and

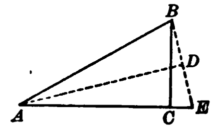


FIG. 34

$$AE = AB = c, \quad CE = c - b.$$

Consequently we find, from the right triangle BCE,

$$(2) \qquad \tan \tfrac{1}{2} A = \frac{CE}{BC} = \frac{c-b}{a}.$$

But

$$a = \sqrt{c^2 - b^2} = \sqrt{(c-b)(c+b)},$$

so that we may write, in place of (2),

$$(3) \qquad \tan \tfrac{1}{2} A = \sqrt{\frac{c-b}{c+b}}.$$

This is the desired formula, which should be applied instead of (1), whenever the value of b is very close to that of c, *i.e.* whenever the angle A is very small.

Some of the following examples will illustrate the usefulness of this formula as well as the application of Table III for the functions of small angles.

EXERCISE XVII

1. At what distance may a mountain 14,000 feet high be seen at sea, if the earth's radius is 3963 miles?

2. How high above the earth's surface must a balloon rise, in order to enable an observer to see a point 50 miles away?

3. If the moon's parallax (the angle which the earth's radius subtends as seen from the moon) is 57′, and if the earth's radius is 3963 miles, what is the moon's distance from the earth?

4. If the angular diameter of the moon, as seen from the earth, is 31′ 20″ and the distance from the earth to the moon is 239,100 miles, what is the moon's diameter in miles?

5. If the distance from the earth to the sun is 92,000,000 miles, and the angular diameter of the sun as seen from the earth is 32′, what is the diameter of the sun in miles?

CHAPTER VII

THEORY OF OBLIQUE TRIANGLES

40. The area of an oblique triangle in terms of two of its sides and the included angle.

Let the sides b, c of a triangle and the angle A be given.

If we consider the side $AB = c$ as the base, then the altitude $CD = h$ is the length of the perpendicular dropped from C to AB. The foot D of this perpendicular may fall on the

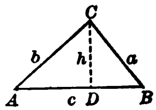


FIG. 35

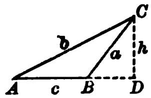


FIG. 36

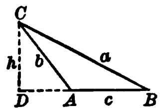


FIG. 37

line segment AB (Fig. 35), to the right of B (Fig. 36), or to the left of A (Fig. 37). In all of these cases we have,

(1) $$S = \tfrac{1}{2}\, ch,$$

if S denotes the area of the triangle.

Now h can be expressed in terms of the given quantities, b, c, and A, as follows.

In Figs. 35 and 36, in which A is an acute angle, we have

$$\frac{h}{b} = \sin A \text{ or } h = b \sin A,$$

and therefore, by substituting this value of h in (1),

(2) $$S = \tfrac{1}{2}\, bc \sin A.$$

If A is an obtuse angle (Fig. 37), we have

$$\frac{h}{b} = \sin \angle DAC = \sin (180^\circ - A), \text{ or } h = b \sin (180^\circ - A),$$

and consequently, by substituting this value of h in (1),

(3) $$S = \tfrac{1}{2}\, bc \sin (180^\circ - A).$$

Thus, the area of a triangle is equal to

$$\tfrac{1}{2}\,bc \sin A \quad \text{or} \quad \tfrac{1}{2}\,bc \sin(180^\circ - A)$$

according as A is an acute or an obtuse angle.

While we have found a complete solution of our problem, the result is not quite as convenient as it might be, since we have two different formulæ for S according as A is an acute or an obtuse angle. Is there any way in which we might avoid the distinction between these two cases?

The expression

$$\tfrac{1}{2}\,bc \sin A$$

is meaningless, from our present point of view, if A is an obtuse angle. For we have, as yet, given no definition for the sine of an obtuse angle, the definitions of Art. 7 being applicable to acute angles only. Clearly, however, it will be desirable to attach a meaning to the symbol $\sin A$, also in the case when A is an obtuse angle, now that we are dealing with oblique triangles, some of whose angles may be obtuse.

We may define the sine of an obtuse angle in any way we choose, *so long as it is not inconsistent with the definitions already agreed upon*, and we naturally choose our definitions and notations in such a way as to reduce to a minimum the number of formulæ and theorems which must be remembered.

Now we can make a single formula do the work of both (2) and (3), by adopting the following **definition for the sine of an obtuse angle.**

The sine of an obtuse angle A is equal to the sine of the acute angle $180^\circ - A$, *which is supplementary to A;* or in symbols,

(4) $\qquad \mathbf{\sin A = \sin(180^\circ - A)}$, **$A$ being obtuse.**

As a consequence of this definition, equation (3), which gives the expression for the area of the triangle when A is obtuse, reduces to

$$S = \tfrac{1}{2}\,bc \sin A,$$

so that formula (2) may be used whether A be acute or obtuse. The same formula is obviously true when A is a

right angle; for in that case $\sin A = 1$, and the formula reduces to

$$S = \tfrac{1}{2}\, bc$$

where c is the base and b the altitude.

We therefore have a single formula

$$S = \tfrac{1}{2}\, bc \sin A$$

for the area of a triangle in terms of two sides and the included angle, whether the latter be acute, right, or obtuse.

41. The law of sines. Since the triangle has three angles and three pairs of including sides, we may write three different expressions for the area of the same triangle, viz.:

$$S = \tfrac{1}{2}\, bc \sin A = \tfrac{1}{2}\, ca \sin B = \tfrac{1}{2}\, ab \sin C.$$

The equality of these three expressions is a very important fact, since it gives rise to the following relations between the sides and angles of any triangle:

$$bc \sin A = ca \sin B = ab \sin C.$$

We may write these relations in a somewhat simpler form, by dividing all three members of the continued equation by the product abc. We find in this way

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

or

(1) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

whence

(2) $$\frac{a}{b} = \frac{\sin A}{\sin B},\quad \frac{a}{c} = \frac{\sin A}{\sin C},\quad \frac{b}{c} = \frac{\sin B}{\sin C}.$$

These formulæ contain the so-called **law of sines,** which may be expressed in words as follows: *any two sides of a triangle are to each other as the sines of the opposite angles.*

The first explicit statement and proof of the law of sines, known at the present day, is to be found in a treatise on trigonometry by the Persian, NASĪR ADDĪN, or NASIR EDDIN (1201–1247 A.D.). Nasīr Addīn's treatise may also be regarded as the first in which trigonometry was treated as a separate science, independent of its applications to astronomy.

EXERCISE XVIII

1. What becomes of the law of sines when one of the angles (say C) is a right angle?

2. Prove the law of sines directly from Figs. 35, 36, 37, by computing the value of h in each of the two right triangles into which ABC is divided by the altitude.

3. Show that the law of sines may be used to solve the following problem: Given two angles of a triangle and one of its sides; to find the other sides and the remaining angle.

4. The formula,
$$\sin(180° - A) = \sin A,$$
holds when A is an obtuse angle, as a consequence of the definition adopted for the sine of an obtuse angle. Show that the same equation is also true if A is an acute or right angle.

42. The law of cosines. A generalization of the theorem of Pythagoras. We have learned to recognize the importance of the theorem of Pythagoras in the theory of right triangles, and the question naturally arises: what takes the place of this theorem in the case of an oblique triangle?

Most students will remember that the answer to this question is contained in the following two propositions of geometry:

Theorem 1. In any triangle the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides *diminished* by twice the product of one of those sides and the projection of the other upon that side.

Theorem 2. In any obtuse triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides *increased* by twice the product of one of those sides and the projection of the other upon that side.

The proof of Theorem 1 (repeated from Geometry) is as follows: Let A be an acute angle. The triangle ABC will have the form represented in Figs. 38 or 39, accord-

ing as the angle B is acute or obtuse. In either case we put

$$BC = a, \quad CA = b, \quad AB = c,$$

and

$$AD = m,$$

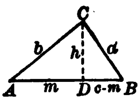


FIG. 38

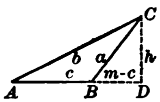


FIG. 39

so that m is the projection of b upon c, or upon c produced. The right triangle BCD gives

$$a^2 = h^2 + \overline{BD}^2$$

in both cases. Now $BD = c - m$ in Fig. 38, and $BD = m - c$ in Fig. 39. Therefore we find, in either case,

$$a^2 = h^2 + c^2 - 2\,cm + m^2.$$

The right triangle ACD gives

$$h^2 = b^2 - m^2$$

in both cases. If this value of h^2 be substituted in the equation above, we find

(1) $$a^2 = b^2 + c^2 - 2\,cm \text{ } (\textit{A being an acute angle}),$$

which proves Theorem 1.

To prove Theorem 2, we refer to Fig. 40. We have, in this case,

$$a^2 = h^2 + \overline{BD}^2 = h^2 + (c + m)^2$$

or

$$a^2 = h^2 + c^2 + 2\,cm + m^2,$$

and

$$h^2 = b^2 - m^2,$$

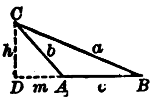


FIG. 40

whence

(2) $$a^2 = b^2 + c^2 + 2\,cm \text{ } (\textit{A being an obtuse angle}),$$

which proves Theorem 2.

From either Fig. 38 or 39 we obtain, by observing the right triangle ACD,

$$m = AD = b \cos A.$$

In Fig. 40 we have instead,

$$\angle\, CAD = 180^\circ - A, \quad m = b \cos CAD = b \cos(180^\circ - A).$$

If we substitute these values in (1) and (2), we find

(3) $$a^2 = b^2 + c^2 - 2\,bc \cos A \text{ } (\textit{if A is acute}),$$

(4) $$a^2 = b^2 + c^2 + 2\,bc \cos(180^\circ - A) \text{ } (\textit{if A is obtuse}).$$

Just as in Art. 41, we have found two different theorems and two different formulæ for the two cases when A is an acute or an obtuse angle. Can we again find a *single* theorem and a *single* formula to do the work of both?

Equations (3) and (4) show that this may indeed be done, provided that *we define the cosine of an obtuse angle to be a negative number, numerically equal to the cosine of its supplement* (which is of course an acute angle). For, with this definition, we shall have

(5) $\cos A = -\cos(180° - A)$ (*if A is an obtuse angle*),

so that (4), as a consequence of (5), assumes the same form as (3). But formula (3) holds also when A is a right angle, for in that case $\cos A = 0$, and the formula reduces to the theorem of Pythagoras. Thus, *one and the same formula* (3)

$$a^2 = b^2 + c^2 - 2\,bc \cos A$$

will be applicable to all cases if it be understood, in accordance with our definitions, that $\cos A$ is positive, zero, or negative according as the angle A is acute, right, or obtuse.

Equation (3) is generally known as the **law of cosines**, and completely replaces Theorems 1 and 2 of this Article. The law of cosines obviously enables us to compute the third side of an oblique triangle when two sides and the included angle are given. But it also enables us to find the angles of a triangle when its three sides are given. For, we find from (3), by transposition,

$$2\,bc \cos A = b^2 + c^2 - a^2,$$

and therefore

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,bc}. \tag{6}$$

The two geometric theorems (Theorems 1 and 2 of this article) to which the law of cosines is equivalent, were well known to the Ancients; and the problem of finding the angles of a triangle, when its three sides are given, was solved by Ptolemy (2d century A.D.) of Alexandria in his *Almagest* by means of these theorems. The explicit formulation of the law of cosines, however, seems to be due to the great French mathematician, François Viète (also known as Vieta), (1540–1603).

EXERCISE XIX

Solve the following triangles, using the tables of squares and natural functions:

1. $a = 2,\quad b = 3,\quad C = 30°.$

2. $b = 3.5,\ c = 2.4,\ A = 52°.$

3. $c = 2.34,\ a = 4.31,\ B = 116°.$

4. $a = 3,\quad b = 6,\quad c = 8.$

5. $a = 1.0,\quad b = 2.0,\quad c = 1.5.$

6. The relation $\cos A = -\cos(180° - A)$ is true for all obtuse angles A as a consequence of the definition of the cosine of an obtuse angle. Prove that this formula is also true if A is any acute angle or a right angle.

7. Show that the relation $\sin^2 A + \cos^2 A = 1$ holds for obtuse as well as for acute angles.

8. If A is an acute angle,

$$\tan A = \frac{\sin A}{\cos A},\ \cot A = \frac{\cos A}{\sin A},\ \sec A = \frac{1}{\cos A},\ \csc A = \frac{1}{\sin A}.$$

Let us *define* $\tan A$, $\cot A$, $\sec A$, $\csc A$ by means of these same equations when A is an obtuse angle. Show that, as a consequence of these definitions, we have

$$\tan A = -\tan(180° - A),\quad \cot A = -\cot(180° - A),$$
$$\sec A = -\sec(180° - A),\quad \csc A = \csc(180° - A),$$

for any obtuse angle A.

9. Show that the equations of Ex. 8 are valid also if the angle A is acute.

10. The law of cosines gives three equations for any triangle. Equation (3) of Art. 42 is one of these. Write the other two.

11. Write the equations for $\cos A$, $\cos B$, and $\cos C$ in terms of the three sides of the triangle.

12. Show that in any triangle,

$$a^2 + b^2 + c^2 - 2\,bc\cos A - 2\,ca\cos B - 2\,ab\cos C = 0.$$

43. Properties of the functions of an obtuse angle.* We have defined in Art. 40 the sine, in Art. 42, the cosine, and in Ex. 8, Exercise XIX, the remaining functions of an obtuse angle. As a consequence of these definitions we have the following system of equations :

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* Some instructors may prefer to change the order of topics by passing to the discussion of the general angle, Art. 60 *et seq.*, and returning later to Art. 44. There is no reason why this should not be done.

$$(1)\quad \begin{cases} \sin(180^\circ - A) = \sin A, & \cos(180^\circ - A) = -\cos A, \\ \tan(180^\circ - A) = -\tan A, & \cot(180^\circ - A) = -\cot A, \\ \sec(180^\circ - A) = -\sec A, & \csc(180^\circ - A) = \csc A, \end{cases}$$

which are valid whether the angle A be acute, right, or obtuse. (Cf. Exercise XVIII, Ex. 4, Exercise XIX, Exs. 6, 8, and 9.)

But an obtuse angle B may be written either in the form

$$B = 180^\circ - A, \text{ or } B = 90^\circ + A',$$

both A and A' being acute angles. Moreover, from

$$B = 180^\circ - A = 90^\circ + A'$$

follows

$$A' = 90^\circ - A, \text{ or } A = 90^\circ - A',$$

that is, the angles A and A' are complementary. Let us substitute $A = 90^\circ - A'$ in (1). We find that the left member of the first equation of (1) becomes

$$\sin(180^\circ - A) = \sin[180^\circ - (90^\circ - A')] = \sin(90^\circ + A').$$

The right member of the same equation assumes the form

$$\sin A = \sin(90^\circ - A') = \cos A'. \qquad \text{(Art. 10.)}$$

Consequently, this first equation of system (1) becomes

$$\sin(90^\circ + A') = \cos A'.$$

In the same way we may prove the remaining equations of the following system:

$$(2)\quad \begin{cases} \sin(90^\circ + A') = \cos A', & \cos(90^\circ + A') = -\sin A', \\ \tan(90^\circ + A') = -\cot A', & \cot(90^\circ + A') = -\tan A', \\ \sec(90^\circ + A') = -\csc A', & \csc(90^\circ + A') = \sec A'. \end{cases}$$

The student should carry out the details of these substitutions, and note the close resemblance between these formulæ and the formulæ of Art. 10, for $\sin(90^\circ - A)$, $\cos(90^\circ - A)$, etc.

This resemblance is so close as to make it easy to remember equations (2). Their right members differ from the right members of the corresponding equations for $\sin(90^\circ - A)$, $\cos(90^\circ - A)$, etc., at most in sign. This remark suffices to help us remember that $\sin(90^\circ + A)$

is either equal to $+\cos A$ or to $-\cos A$, that $\cos(90^\circ + A)$ is either equal to $+\sin A$ or $-\sin A$, etc. In order to choose between these alternatives we may argue as follows. Let A be an acute angle. Then $\sin A$ and $\cos A$ are both positive. But since $90^\circ + A$ will then be obtuse, $\sin(90^\circ + A)$ is positive and $\cos(90 + A)$ is negative. Therefore, of the two alternatives

$$\sin(90^\circ + A) = \cos A, \text{ or } \sin(90^\circ + A) = -\cos A,$$

we must discard the latter, since its left member is positive and its right member is negative. Similarly, of the two alternatives

$$\cos(90^\circ + A) = \sin A, \text{ or } \cos(90 + A) = -\sin A,$$

we must discard the former. For it involves the contradiction that the negative number $\cos(90^\circ + A)$ should be equal to the positive number $\sin A$. This argument fixes the two formulæ

$$\sin(90^\circ + A) = \cos A, \quad \cos(90^\circ + A) = -\sin A$$

in our memory. The remaining formulæ of system (2) may be remembered in similar fashion.

The first two formulæ of system (1), namely,

$$\sin(180^\circ - A) = \sin A, \quad \cos(180^\circ - A) = -\cos A,$$

are easy to remember, since it was upon these equations that we based our definitions of the sine and cosine of an obtuse angle. The remaining formulæ of system (1) follow directly from these two.

EXERCISE XX

1. Express the following quantities as functions of acute angles; $\sin 100^\circ$, $\cos 115^\circ$, $\tan 162^\circ$, $\cot 99^\circ$, $\sec 120^\circ$, $\csc 175^\circ$.

2. By means of the table of natural functions, find $\sin 98^\circ.5$, $\cos 176^\circ.3$, $\tan 124^\circ.7$, $\cot 134^\circ.6$.

3. Explain how equations (1) and (2) of Art. 43 provide two different methods of finding the functions of obtuse angles from the table for acute angles.

44. Other formulæ for the area of an oblique triangle. The methods of Art. 40 suffice to determine the area of a triangle if one side and the corresponding altitude (c and h), or if two sides and the included angle (b, c, and A) are given.

Let us suppose now that one side and two adjacent angles

(c, A, B) are given. We have in the first place from Art. 40

(1) $$S = \tfrac{1}{2} bc \sin A,$$

and it only remains to modify this formula by expressing b in terms of c, A, and B. The law of sines, in the form

(2) $$\frac{b}{c} = \frac{\sin B}{\sin C},$$

and the equation

(3) $$A + B + C = 180°$$

enable us to do this. For we find from (2) and (3)

$$b = \frac{c \sin B}{\sin C} = \frac{c \sin B}{\sin [180° - (A+B)]} = \frac{c \sin B}{\sin (A + B)}$$

since, according to Art. 43, the sine of any acute or obtuse angle is equal to the sine of its supplement.

If we substitute this value of b in (1), we find

(4) $$S = \frac{c^2 \sin A \sin B}{2 \sin (A + B)},$$

the desired formula for S in terms of c, A, and B.

If one side c and two angles, not both adjacent to c, are given, we may first find the third angle from (3) and then use formula (4).

Let us, finally, obtain a formula for S in terms of the three sides a, b, c. We start again from equation (1) which already contains the sides b and c of the triangle, and which will give the desired formula if we can express $\sin A$ in terms of a, b, and c. This may be done by making use of the law of cosines (which gives $\cos A$ in terms of a, b, and c), and the relation

(5) $$\sin^2 A + \cos^2 A = 1,$$

which holds for obtuse as well as for acute angles. (See Ex. 7, Exercise XIX.)

To avoid the inconvenience of writing cumbersome square

root signs, let us square both members of (1). We find, making use of (5),

$$S^2 = \frac{1}{4} b^2c^2 \sin^2 A = \frac{1}{4} b^2c^2 (1 - \cos^2 A)$$

or, factoring the binomial on the right,

$$S^2 = \frac{1}{4} b^2c^2 (1 + \cos A)(1 - \cos A).$$

By the law of cosines (Equation (6), Art. 42), this becomes

$$\begin{aligned} S^2 &= \frac{1}{4} b^2c^2 \left(1 + \frac{b^2 + c^2 - a^2}{2\,bc}\right)\left(1 - \frac{b^2 + c^2 - a^2}{2\,bc}\right) \\ &= \frac{b^2c^2}{4} \cdot \frac{2\,bc + b^2 + c^2 - a^2}{2\,bc} \cdot \frac{2\,bc - b^2 - c^2 + a^2}{2\,bc} \\ &= \frac{1}{16}\left[(b + c)^2 - a^2\right]\left[a^2 - (b - c)^2\right]. \end{aligned}$$

Each of the factors on the right member may be factored, giving the equation

(6) $S^2 = \frac{1}{16}(b + c + a)(b + c - a)(a + b - c)(a - b + c).$

The first factor on the right member of (6) is the perimeter of the triangle, and the formula becomes especially simple if we denote half of this perimeter by s, so that

(7) $$a + b + c = 2\,s.$$

The other three factors of the right member of (6) will then become

(8) $$\begin{aligned} -a + b + c &= 2\,s - 2\,a = 2\,(s - a), \\ a - b + c &= 2\,s - 2\,b = 2\,(s - b), \\ a + b - c &= 2\,s - 2\,c = 2\,(s - c), \end{aligned}$$

so that (6) reduces to

$$S^2 = s\,(s - a)(s - b)(s - c),$$

whence finally, since S is positive,

(9) $$S = \sqrt{s\,(s - a)(s - b)(s - c)},$$

a famous equation generally known as **Hero's formula**, after Hero of Alexandria, who lived about 120 B.C., and wrote a famous textbook on surveying.

EXERCISE XXI

Find the area of the following triangles to four significant figures.

1. $a = 2, \quad b = 3, \quad C = 30°.$
2. $b = 3.5, c = 2.4, A = 52°.$
3. $c = 5, \quad A = 30°, B = 75°.$
4. $a = 3, \quad A = 30°, B = 75°.$
5. $a = 3, \quad b = 6, \quad c = 8.$
6. $a = 1.0, b = 2.0, \quad c = 1.5.$

45. The radius and center of the inscribed circle. The fundamental basis for all of the different formulæ which we have obtained for the area of a triangle, so far, was the equation

$$S = \tfrac{1}{2}\, ch,$$

from which all of the others were derived.

But this formula is unsymmetrical, since it singles out one of the sides of the triangle and subjects it to a treatment different from that accorded to the other two. We may avoid this lack of symmetry by picking out a point M anywhere inside of the triangle ABC (see Fig. 41), and joining M to the three vertices. The area of the given triangle will then appear as the sum of the areas of the three triangles BMC, CMA, AMB, whose altitudes are respectively equal to MD, ME, and MF.

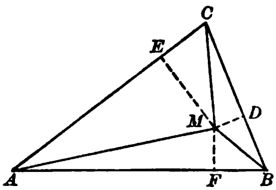

FIG. 41

Clearly, there is one position of the point M which is better adapted for this purpose than any other, namely the center O of the inscribed circle. For the distances from O to the three sides of the triangle are equal to each other, so that the three triangles BOC, COA, and AOB will have equal altitudes.

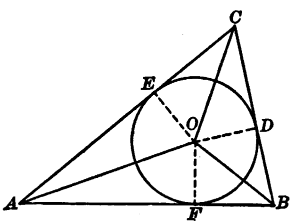

FIG. 42

Let O (Fig. 42) be the center of the inscribed circle and let r be the radius of this circle.

Then the area of the triangle is

$$\begin{aligned} S &= BOC + COA + AOB \\ &= \tfrac{1}{2}\, ar + \tfrac{1}{2}\, br + \tfrac{1}{2}\, cr \\ &= \tfrac{1}{2}\,(a + b + c)\, r, \end{aligned}$$

or finally

(1) $$S = sr,$$

where s denotes the half-perimeter of the triangle as in Art. 44.

Since we also have

$$S = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{(Equation (9), Art. 44)},$$

we find from (1), substituting this value of S and dividing both members of the resulting equation by s,

(2) $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

which enables us to compute the radius of the inscribed circle when the sides of the triangle are given.

If we wish to know, not merely the *radius* of the inscribed circle, but also the position of its center, we must compute the lengths of the line segments AF, BF, etc. in Fig. 42. Now we know from Geometry that

(3) $$AF = AE, \; BD = BF, \; CE = CD.$$

Moreover, the sum of these six line-segments is equal to the perimeter $2s$ of the triangle. Therefore the sum of three of these segments, one chosen from each of the three equations (3), will be equal to half of the perimeter. That is,

$$AF + BD + CD = s.$$

But

$$BD + CD = a,$$

and therefore

(4) $$AF = AE = s - a.$$

In the same way we find

(5) $$\begin{cases} BD = BF = s - b, \\ CE = CD = s - c. \end{cases}$$

The three equations, (4) and (5), enable us to locate the points D, E, F in which the inscribed circle touches the sides of the triangle and consequently determine completely the position of the center O of the circle.

46. The half-angle formulæ. Since the center O of the inscribed circle is the point of intersection of the three angle bisectors of the triangle (see Fig. 42), the angle FAO is equal to $\frac{1}{2}A$. In the right triangle FAO, we have therefore

$$\tan \tfrac{1}{2} A = \frac{FO}{AF}.$$

But FO is the radius r of the inscribed circle, and AF we have just found to be equal to $s - a$. (Equation (4), Art. 45.) We find therefore

$$(1) \qquad \tan \tfrac{1}{2} A = \frac{r}{s-a},$$

and in the same way

$$(2) \qquad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c},$$

the value of r being given by equation (2) of Art. 45.

These three equations, usually known as **the half-angle formulæ**, are very important in providing a second method for computing the angles of a triangle when its sides are given. They are far more convenient for this purpose than the law of cosines, if logarithms are to be used. In fact, the law of cosines is so cumbersome from the point of view of logarithmic calculation, that we shall seek to find a substitute for it also in the only other case in which we have proposed to make any use of it, namely in the solution of a triangle when two sides and the included angle are given. (See Art. 42, p. 96.)

We should not neglect to note, however, that the law of cosines has in recent times again come into practical use, especially in engineering practice, the calculations being performed not with logarithms, but with the help of tables of squares and products or with a calculating machine. The Eichhorn Trigonometric slide rule, mentioned on page 66, is based entirely upon the law of cosines.

47. The circumscribed circle. The center O of the circumscribed circle is the point of intersection of the three perpendicular bisectors of the sides of the triangle. Therefore,

if N is the middle point of the side AB of the triangle (see Fig. 43), the line ON will be perpendicular to AB.

Now angle AOB is measured by the arc AB, and the angle ACB is measured by half this arc (being an inscribed angle).

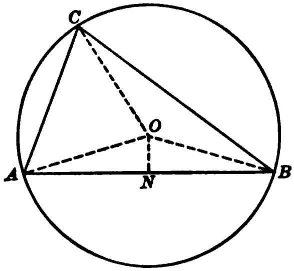

FIG. 43

Therefore

$$\angle AON = \angle ACB = C$$

since each of these angles is equal to half of $\angle AOB$. Consequently we find, making use of the right triangle AON,

$$\sin C = \sin AON = \frac{AN}{AO}.$$

If we denote AO, the radius of the circumscribed circle, by R, this becomes

$$\sin C = \frac{\frac{1}{2}c}{R} = \frac{c}{2R}, \text{ or } 2R = \frac{c}{\sin C}$$

whence, making use of the law of sines (Equation (1), Art. 41),

(1) $$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

thus completing the law of sines by providing a simple geometrical interpretation for the common value of the three equal ratios

$$\frac{a}{\sin A}, \frac{b}{\sin B}, \frac{c}{\sin C};$$

namely, the diameter of the circumscribed circle.

Since the area of the triangle is

$$S = \tfrac{1}{2} bc \sin A,$$

and since we find, from (1),

$$\sin A = \frac{a}{2R},$$

we obtain the following remarkable formula

(2) $$S = \frac{abc}{4R}$$

for the area of the triangle.

EXERCISE XXII

1. Compute the radius of the inscribed circle and the lengths of the line-segments into which this circle divides the sides of the triangle whose sides are $a = 3$, $b = 6$, $c = 8$.

2. What relation will there be between the sides of a triangle if the inscribed circle bisects one of its sides? If it touches one of the sides at a trisection point?

3. Find formulæ for the distances from the center of the inscribed circle to the three vertices of the triangle.

4. Making use of the results of Ex. 3, show that

$$\sin \tfrac{1}{2} A = \sqrt{(s-b)(s-c)/bc} \text{ and } \cos \tfrac{1}{2} A = \sqrt{s(s-a)/bc}.$$

5. Find a formula for the area of the inscribed circle, and for the ratio of this area to that of the triangle itself.

6. By means of the formulæ of Ex. 5, show that

$$\pi \sqrt{\frac{(s-a)\ (s-b)\ (s-c)}{s^3}} < 1.$$

7. Show that

$$r = s \tan \tfrac{1}{2} A \tan \tfrac{1}{2} B \tan \tfrac{1}{2} C.$$

8. A circle is said to be *escribed to a triangle*, or inscribed externally, if it touches one of the sides externally and the prolongations of the other two sides (Fig. 44). There are three such circles for every triangle. Let r_a be the radius of that one which touches the side a externally. Show that

$$AM = AN = s$$

and

$$r_a = s \tan \tfrac{1}{2} A = \frac{S}{s-a} = \frac{rs}{s-a}.$$

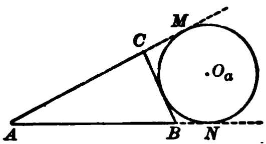

FIG. 44

9. If r_a, r_b, r_c, are the radii of the three escribed circles, and r denotes the radius of the inscribed circle, show that

$$\frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}.$$

10. Show that

$$r_a + r_b + r_c - r = 4R,$$

if R is the radius of the circumscribed circle.

48. The form ratios of a triangle. If two sides of a triangle are equal, the triangle is said to be *isosceles;* if no two sides are equal, it is usually called a *scalene* triangle. But if the difference between two sides of a triangle is small,

as compared with the combined length of these two sides, the triangle will differ but little from the isosceles form. Consequently, the ratio of the difference between two sides of a triangle to their combined length may be taken as a numerical measure for the *departure of the triangle from the isosceles* form, or as its **form ratio** with respect to those two sides.

There are three such form ratios for every triangle. If we assume that the notation be so chosen that

$$a \geqq b \geqq c,$$

these three form ratios are

$$\frac{a-b}{a+b}, \quad \frac{a-c}{a+c}, \quad \frac{b-c}{b+c}.$$

Clearly, one of them will be equal to zero if the triangle is isosceles, and all three will vanish for an equilateral triangle.

By means of the law of sines, each of these form ratios may be expressed in terms of the angles of the triangle. In fact, according to Art. 47, equation (1), we have

$$a = 2\,R\sin A, \quad b = 2\,R\sin B, \quad c = 2\,R\sin C,$$

so that we obtain the expression

$$(1) \qquad \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

for the form ratio with respect to the sides a and b. Similar expressions hold, of course, for the other form ratios; $\frac{a-c}{a+c}$ and $\frac{b-c}{b+c}$.

49. The formulæ for the sum and difference of two sines. The fraction which occurs in the right-hand member of equation (1), Art. 48, is not adapted to logarithmic calculation. But we shall show that both numerator and denominator of this fraction, namely, the expressions $\sin A - \sin B$, and $\sin A + \sin B$, may be written in the form of products, thus making it easy to compute their values by logarithms.

To discover the product form of these expressions, we must construct a figure in which the angles A and B, the sines of these angles, and the sum and difference of their sines, shall appear.

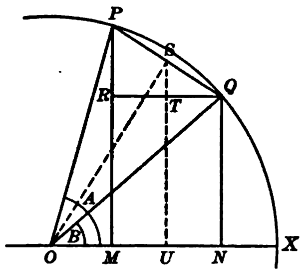

FIG. 45

We construct first (Fig. 45) the angles

$$\angle XOP = A \text{ and } \angle XOQ = B.$$

With their common vertex O as center, and any convenient radius r, we draw a circle whose intersections with the sides of the angles we denote by X, P, and Q, respectively.

Let PM and QN be perpendicular, and QR parallel to OX. Then

$$(1) \qquad r \sin A = MP, \text{ and } r \sin B = NQ.$$

But MP and NQ are the two bases of the trapezoid $MPQN$. If S is the middle point of the chord PQ, and US be drawn perpendicular to OX, US is the median of this trapezoid, and we shall have,

$$US = \tfrac{1}{2}(MP + NQ),$$

whence, according to (1),

$$(2) \qquad r(\sin A + \sin B) = MP + NQ = 2\,US.$$

On the other hand,

$$TS = \tfrac{1}{2}PR = \tfrac{1}{2}(MP - NQ),$$

and hence,

$$(3) \qquad r(\sin A - \sin B) = MP - NQ = 2\,TS.$$

In order to express the right members of (2) and (3) in terms of r, A and B, we consider the right triangles OSU and SQT in which US and TS occur. It is apparent from the figure that

$$(4) \qquad \begin{cases} \angle SOQ = \tfrac{1}{2}\, QOP = \tfrac{1}{2}(A - B), \\ \angle UOS = B + \tfrac{1}{2}(A - B) = \tfrac{1}{2}(A + B), \end{cases}$$

and

$$(5) \qquad \angle TSQ = \angle UOS = \tfrac{1}{2}(A + B),$$

since each of these angles is complementary to the angle USO.

Hence we have from the right triangles OSU and SQT,

$$(6) \qquad \begin{aligned} US &= OS \sin UOS = OS \sin \tfrac{1}{2}(A + B), \\ TS &= SQ \cos TSQ = SQ \cos \tfrac{1}{2}(A + B), \end{aligned}$$

and from the triangle OQS,

$$(7) \qquad \begin{aligned} OS &= OQ \cos SOQ = r \cos \tfrac{1}{2}(A - B), \\ SQ &= OQ \sin SOQ = r \sin \tfrac{1}{2}(A - B). \end{aligned}$$

Substituting (6) and (7) in (2) and (3) and dividing by r, we find

$$(8) \qquad \begin{aligned} \sin A + \sin B &= 2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B), \\ \sin A - \sin B &= 2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B), \end{aligned}$$

which are the desired formulæ.

The student should observe that our proof of equations (8), based on Fig. 45, remains valid even if A is an obtuse angle, as long as MP is greater than or equal to NQ; that is, as long as $A + B$ does not exceed 180°. We may therefore apply equations (8) whenever A and B are two angles of the same triangle, since the sum of two angles of a triangle can never exceed 180°.

50. The law of tangents. Let us now return to the expression (1) of Art. 48 for the form ratio of the triangle with respect to the sides a and b, of which sides we shall assume a to be the greater. We had found

$$\frac{a - b}{a + b} = \frac{\sin A - \sin B}{\sin A + \sin B}.$$

If now we make use of equations (8) of Art 49, we find

$$\frac{a - b}{a + b} = \frac{2 \cos \tfrac{1}{2}(A + B) \sin \tfrac{1}{2}(A - B)}{2 \sin \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)},$$

whence

$$\frac{a-b}{a+b} = \cot \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B),$$

since we have, for any angle M,

$$\frac{\sin M}{\cos M} = \tan M, \quad \frac{\cos M}{\sin M} = \cot M.$$

But the tangent and cotangent of the same angle are reciprocals, so that we may write finally

$$(1) \qquad \frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)},$$

a very important formula, generally known as **the law of tangents.** We find in the same way:

$$(2) \qquad \begin{cases} \dfrac{a-c}{a+c} = \dfrac{\tan \frac{1}{2}(A-C)}{\tan \frac{1}{2}(A+C)}, \\ \dfrac{b-c}{b+c} = \dfrac{\tan \frac{1}{2}(B-C)}{\tan \frac{1}{2}(B+C)}. \end{cases}$$

The law of tangents seems to have been expressed in this form for the first time by VIETA, to whom we also owe the modern form of the law of cosines. It had however been stated, in a more complicated but equivalent form, about ten years earlier by the Dutch mathematician THOMAS FINK or FINCHIUS (1561–1656) in his *Geometria rotundi.* It was also Fink who first introduced the names tangent and secant for the functions which we now call by these names. Many previous authors had used the name *umbra* = shadow for the function which we now call tangent, on account of the relation of this function to the shadow cast by a vertical stick. (Compare Art. 3 and solve the problem there attributed to Thales by trigonometry.)

Fink not only discovered the law of tangents, but pointed out its principal application; namely, to aid in solving a triangle when two sides and the included angle are given. The possibility of such an application will appear from the following

Illustrative Example. Given a, b, and C. To find A, B, and C.
Solution. The law of tangents (Equation (1)) gives

$$(3) \qquad \tan \tfrac{1}{2}(A-B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A+B),$$

an equation in which the right member is completely known, since a, b, and C are given, and since

$$(4) \qquad \tfrac{1}{2}(A + B) = \tfrac{1}{2}(180° - C) = 90° - \tfrac{1}{2} C.$$

Thus we can compute $\tan \frac{1}{2}(A - B)$ by means of (3) and then find $\frac{1}{2}(A - B)$ from the table. Hence, knowing $\frac{1}{2}(A - B)$ and $\frac{1}{2}(A + B)$ we can find

$$(5) \qquad \begin{aligned} A &= \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B), \\ B &= \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B). \end{aligned}$$

The sine law now enables us to find c, since

$$(6) \qquad \frac{c}{a} = \frac{\sin C}{\sin A} \text{ gives } c = \frac{a \sin C}{\sin A}.$$

If we wish to solve the same problem by the law of cosines, we first compute c from the equation

$$(7) \qquad c^2 = a^2 + b^2 - 2ab \cos C$$

and afterward determine the angles A and B by using the law of sines.

The first method has the advantage over the second that formulæ (3), (4), (5), (6) are in a convenient form for logarithmic computation, while equation (7) is not.

51. A second proof of the law of tangents and Mollweide's equations. The law of tangents may also be proved directly from a figure without making use of the formulæ for

$$\sin A + \sin B$$

and

$$\sin A - \sin B.$$

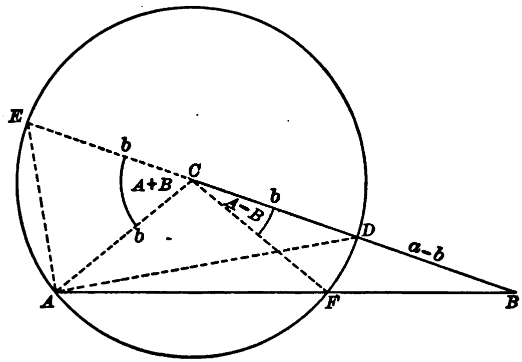

FIG. 46

Let $BC = a$ and $CA = b$ (Fig. 46) be two sides of the triangle ABC, and let $a > b$. Draw a circumference with C as center and radius equal to b, and let D and E be the points in which this circumference meets BC and BC produced. Let F be the point of intersection of the circumference with AB. Then we have

(1) $$BD = a - b,\ BE = a + b.$$

Since $\angle ECA$ is an exterior angle of the triangle ABC, the opposite interior angles being A and B, we have

$$\angle ECA = A + B,$$

so that

(2) $$\angle EDA = \tfrac{1}{2}(A + B)$$

since the latter angle, being inscribed in the circumference, is measured by half the arc which it subtends.

It remains to find an angle in Fig. 46, equal to $\frac{1}{2}(A - B)$. In order to do this, let us draw CF. Since ACF is an isosceles triangle, we have

$$\angle AFC = \angle FAC = A$$

and, since $\angle AFC$ is an exterior angle of the triangle BFC,

$$A = \angle FCB + B,$$

whence

$$\angle FCB = A - B.$$

Since $\angle DAF$ is an inscribed angle subtending the same arc,

(3) $$\angle DAB = \angle DAF = \tfrac{1}{2}(A - B).$$

Let us apply the law of sines to the two triangles ABD and ABE. We find, from the first triangle,

$$\frac{a-b}{c} = \frac{\sin DAB}{\sin ADB} = \frac{\sin \frac{1}{2}(A-B)}{\sin(180^\circ - EDA)} = \frac{\sin \frac{1}{2}(A-B)}{\sin EDA}$$

or finally

(4) $$\frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}.$$

Similarly the triangle ABE gives

$$\frac{a+b}{c} = \frac{\sin EAB}{\sin AEB}.$$

But

$$\angle EAB = \angle EAD + \angle DAB = 90^\circ + \tfrac{1}{2}(A - B),$$
$$\angle AEB = 90^\circ - \angle EDA = 90^\circ - \tfrac{1}{2}(A + B),$$

owing to (2) and (3) and the further fact that $\angle EAD$ is a right angle, since it is inscribed in a semi-circumference.

Therefore

$$\frac{a+b}{c}=\frac{\sin[90°+\frac{1}{2}(A-B)]}{\sin[90°-\frac{1}{2}(A+B)]},$$

which may be written

$$(5)\qquad \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)},$$

as a consequence of equations (2), Art. 43, and of Art. 10.

If now we divide equations (4) and (5), member by member, we find

$$\frac{a-b}{a+b}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\,\frac{\cos\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)};$$

that is, the law of tangents.

Incidentally we have found two new formulæ, (4) and (5). They assume a somewhat more serviceable form by means of the relation $A+B+C=180°$, which gives

$$\tfrac{1}{2}(A+B)=90°-\tfrac{1}{2}C,$$

and therefore

$$\sin\tfrac{1}{2}(A+B)=\cos\tfrac{1}{2}C,\quad \cos\tfrac{1}{2}(A+B)=\sin\tfrac{1}{2}C.$$

If we introduce these values in equations (4) and (5), they become

$$(6)\qquad \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C},\quad \frac{a+b}{c}=\frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

These formulæ, known as the **Mollweide equations,** *are particularly convenient for the purpose of checking the accuracy of the numerical solution of a triangle.* For each of these equations contains all of the six parts of the triangle, so that an error in any one of these parts would be likely to make itself felt by a lack of agreement between the two members of one of these equations.

Of course there are two other pairs of equations of the form (6), the left members of which are $\frac{a-c}{b}$, $\frac{a+c}{b}$, and $\frac{b-c}{a}$, $\frac{b+c}{a}$, respectively.

It is not justifiable historically to call equations (6) Mollweide's equations. The formula for $\frac{a+b}{c}$ is to be found in NEWTON's *Arithmetica Universalis.* Both equations (6) are given in SIMPSON's *Trigonometry, Plane and Spherical* (1748), and also in F. W. VON OPPEL's *Analysis Triangulorum* (1746). All of these works antedate considerably the publication of these equations by MOLLWEIDE in 1808.

EXERCISE XXIII

1. Show that the law of tangents may also be obtained from Fig. 46 by drawing through D a line parallel to AE and meeting the side AB in G, and then computing $\tan \frac{1}{2}(A - B)$ and $\tan \frac{1}{2}(A + B)$ from the right triangles ADG and AED.

2. Still another proof of the law of tangents proceeds as follows. In Fig. 47, draw CD, the bisector of angle C, and drop perpendiculars, AL and BM, from A and B to CD. Let N be the point of intersection of AB with CD. Then

$$\angle BCD = \angle DCA = \tfrac{1}{2} C$$

and

$$(1) \qquad \angle LAN = \angle NBM = 90° - \angle BNM.$$

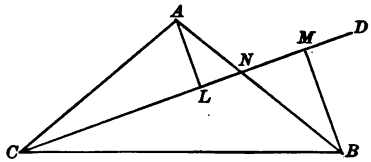

FIG. 47

But $\angle BNM$ is an exterior angle of the triangle BCN, so that

$$\angle BNM = B + \tfrac{1}{2} C.$$

Moreover,

$$90° = \tfrac{1}{2}(A + B + C)$$

and therefore, by substitution in (1),

$$\angle LAN = \tfrac{1}{2}(A + B + C) - (B + \tfrac{1}{2} C) = \tfrac{1}{2}(A - B).$$

Now, from the right triangles ANL and BNM, we see that

$$\tan \tfrac{1}{2}(A - B) = \frac{LN}{LA} = \frac{MN}{MB},$$

and therefore also

$$(2) \qquad \tan \tfrac{1}{2}(A - B) = \frac{LN + MN}{LA + MB}.$$

But

$$LN + MN = CM - CL = a\cos\tfrac{1}{2}C - b\cos\tfrac{1}{2}C = (a-b)\cos\tfrac{1}{2}C,$$

$$LA + MB = b\sin\tfrac{1}{2}C + a\sin\tfrac{1}{2}C = (a+b)\sin\tfrac{1}{2}C,$$

which gives, on substitution in (2),

$$\tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

and this is equivalent to the law of tangents, since

$$C = 180^\circ - (A+B).$$

CHAPTER VIII

SOLUTION OF OBLIQUE TRIANGLES

52. The fundamental problem. Each of the laws found in Chapter VII contains four of the six parts of a triangle, and thus suggests the possibility of computing any one of these four parts when the other three parts which occur in that law are given.

On the other hand, the relation

$$(1) \qquad A + B + C = 180°,$$

which is true for every triangle, contains only three of its parts. The three angles of a triangle are therefore *not independent* of each other, as is the case with the three sides. The familiar fact, that a triangle is not determined by its three angles, is to be regarded as a consequence of this. For, on account of relation (1), the three angles of a triangle are not independent data.

Now *there exists no other equation which, like* (1), *contains no more than three parts of the triangle and which is true for all triangles.* For, if there were such an equation, for instance, between a, b, and c, it would be impossible to find a triangle in which more than two of these quantities have arbitrarily assigned values since two of the three quantities would then determine the third. But this is contrary to well-known facts of geometry; since a triangle may be constructed for which all three sides a, b, and c have arbitrarily assigned values, provided only that $a + b > c$. The values of a and b do *not* determine the value of c; while the values of A and B *do* determine the value of C.

Our illustration shows incidentally that there *may*, and, in fact, *do* exist, besides the *equation* (1), certain *inequalities* between three parts of a triangle which must be satisfied by all triangles. The inequality $a + b > c$ is an instance.

Any three parts of a triangle, unless all three of them are angles, may therefore be regarded as independent data. Consequently there arises the following fundamental problem:

To find the remaining parts of a triangle when any three independent parts are given.

The discussion of this problem leads to a division into four cases.

Case I. One side and two angles are given.

Case II. Two sides and the included angle are given.

Case III. Two sides and the angle opposite to one of them are given.

Case IV. All three sides are given.

53. Case I. Given one side and two angles. Let c, A, B be the given quantities. We may use the formulæ

$$(1) \qquad \begin{cases} C = 180° - (A + B), \\ a = \dfrac{c \sin A}{\sin C}, \quad b = \dfrac{c \sin B}{\sin C}, \end{cases}$$

to compute C, a, and b.

The most reliable and convenient check is furnished by one of Mollweide's equations. (See Equations (6), Art. 51.)

$$(2) \qquad a - b = \frac{c \sin \frac{1}{2}(A - B)}{\cos \frac{1}{2} C}.$$

The law of tangents may also serve as a check instead of (2), but it is not quite as convenient. In calculating the check, it is convenient to think of A as larger than B, so that $A - B$ may be positive. If A is less than B, interchange A and B, and also a and b in equation (2).

EXERCISE XXIV

EXAMPLE 1. Given $c = 327.85$, $A = 110° 52'.9$, $B = 40° 31'.7$. Find C, a, and b. Check the results.

Solution.

$$\text{Given}\begin{cases} c = 327.85 & (1)\\ A = 110^\circ\, 52'.9 & (2)\\ B = 40^\circ\, 31'.7 & (3)\end{cases}$$

$$\begin{aligned} A + B &= 151^\circ\, 24'.6 && (4)\\ C &= 28^\circ\, 35'.4 && (5)\end{aligned}$$

$$\begin{aligned}
\log c &= 2.51568 && (6)\\
\log \sin A &= 9.97049\text{–}10 && (7)\ *\\
\text{colog} \sin C &= 0.32008 && (9)\ \dagger\\
\log a &= 2.80625 && (10)\\
a &= 640.10 && (12)\\
\log c &= 2.51568 && (6)\\
\log \sin B &= 9.81280\text{–}10 && (8)\\
\text{colog} \sin C &= 0.32008 && (9)\\
\log b &= 2.64856 && (11)\\
b &= 445.20 && (13)
\end{aligned}$$

Check.

$$\begin{aligned} A - B &= 70^\circ\, 21'.2 && (14)\\ \tfrac{1}{2}(A - B) &= 35^\circ\, 10'.6 && (15)\\ \tfrac{1}{2} C &= 14^\circ\, 17'.7 && (16)\end{aligned}$$

$$\begin{aligned}
\log c &= 2.51568 && (6)\\
\log \sin \tfrac{1}{2}(A - B) &= 9.76050\text{–}10 && (17)\\
\text{colog} \cos \tfrac{1}{2} C &= 0.01366 && (18)\ \dagger\\
\log(a - b) &= 2.28984 && (19)\\
a - b &= 194.90 && (12)\text{–}(13)\\
a - b &= 194.91 && (\text{From } (19))
\end{aligned}$$

Remarks. The numbers (1), (2), etc., indicate the order in which the separate results are written down and are meant to assist the student in understanding the arrangement of the computation. These numbers should not appear in the student's own work.

Solve and check the following triangles.

2. $a = 467.00$, $A = 56^\circ\, 28'.0$, $B = 69^\circ\, 14'$.
3. $a = 24.31$, $A = 45^\circ\, 18'$, $B = 22^\circ\, 11'$.
4. $a = 148.30$, $A = 37^\circ\, 24'.0$, $C = 76^\circ\, 48'.5$.
5. $A = 71^\circ\, 13'\, 30''$, $B = 40^\circ\, 34'\, 15''$, $c = 236.54$.
6. $a = 3.4356$, $A = 17^\circ\, 43'.4$, $C = 60^\circ\, 35'.7$.
7. $A = 47^\circ\, 13'.2$, $B = 65^\circ\, 24'.5$, $a = 43.176$.
8. $B = 100^\circ\, 21'\, 10''$, $C = 58^\circ\, 17'\, 20''$, $a = 31.656$.
9. $a = 52.780$, $A = 37^\circ 41'\, 15''$, $B = 77^\circ\, 29'\, 40''$.
10. $A = 57^\circ\, 23'\, 12''$, $C = 68^\circ\, 15'\, 30''$, $c = 832.56$.

54. Case II. Given two sides and the included angle. Let a, b, and C be the given parts. If $a > b$, we use the formula

$$\tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2} C \tag{1}$$

* Remember that $\sin 110^\circ\, 52'.9 = \sin (90^\circ + 20^\circ\, 52'.9) = \cos 20^\circ\, 52'.9$. (Art. 43.)
† Remember the rule for finding a cologarithm. (Art. 23.)

to find $\frac{1}{2}(A+B)$, and the law of tangents,

$$(2)\qquad \tan\tfrac{1}{2}(A-B)=\frac{a-b}{a+b}\tan\tfrac{1}{2}(A+B)=\frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

to find $\frac{1}{2}(A-B)$. We then find A and B from

$$(3)\qquad \begin{aligned} A&=\tfrac{1}{2}(A+B)+\tfrac{1}{2}(A-B),\\ B&=\tfrac{1}{2}(A+B)-\tfrac{1}{2}(A-B),\end{aligned}$$

and apply the law of sines to find c, giving

$$(4)\qquad c=\frac{a\sin C}{\sin A}.$$

As checks we use the relations

$$(5)\qquad A+B+C=180°,$$

and one of the Mollweide equations, in the form

$$(6)\qquad \frac{a-b}{c}=\frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

The law of sines furnishes another relation which might be used as a check instead of (6). But (6) is more reliable.

If $a<b$, we interchange the letters a and b, and also A and B in all of the above formulæ, to avoid the appearance of negative angles.

EXERCISE XXV

Example 1. Given $a = 469.71$, $b = 264.37$, $C = 96° 57'.6$. Find A, B, and c and check the results.

Solution.

Given $C = 96° 57'.6$	(1)	$\frac{1}{2}C = 48° 28'.8$	(6)
Given $a = 469.71$	(2)	$\log\cot\frac{1}{2}C = 9.94711 - 10$	(8)
Given $b = 264.37$	(3)	$\log(a-b) = 2.31247$	(10)
$a+b = 734.08$	(4)	$\operatorname{colog}(a+b) = 7.13425 - 10$	(11)
$a-b = 205.34$	(5)	$\log\tan\frac{1}{2}(A-B) = 9.39383 - 10$	(12)
		$\frac{1}{2}(A+B) = 41° 31'.2$	(7)
$\log a = 2.67183$	(18)	$\frac{1}{2}(A-B) = 13° 54'.6$	(13)
$\log\sin C = 9.99679 - 10$	(19)	$A = 55° 25'.8$	(15)
$\operatorname{colog}\sin A = 0.08437$	(20)	$B = 27° 36'.6$	(16)
$\log c = 2.75299$	(21)	$C = 96° 57'.6$	(1)
$c = 566.23$			

Checks.

$\log(a-b) = 2.31247$	(10)	$\log \sin \frac{1}{2}(A-B) = 9.38091 - 10$	(14)
$\log c = 2.75299$	(21)	$\log \cos \frac{1}{2} C = 9.82144 - 10$	(9)
$\log \dfrac{a-b}{c} = 9.55948 - 10$	(22)	$\log \dfrac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2} C} = 9.55947 - 10$	(23)
		$A + B + C = 180° \; 0'.0$	(17)

The checks consist in the result (17) and the close agreement of (22) and (23).

Solve and check the following triangles:

2.	$b = 472$,	$c = 324$,	$A = 78° \, 40'$.
3.	$a = 748$,	$b = 375$,	$C = 63° \, 35'.5$.
4.	$a = 42.38$,	$b = 35.00$,	$C = 43° \, 14' \, 40''$.
5.	$b = 0.941$,	$c = 1.256$,	$A = 35° \, 17' \, 28''$.
6.	$a = 12.3460$,	$b = 5.7213$,	$C = 65° \, 30' \, 10''$.
7.	$a = 25.384$,	$c = 52.925$,	$B = 28° \, 32' \, 20''$.
8.	$b = 0.14367$,	$c = 0.11412$,	$A = 42° \, 14'.6$.
9.	$a = 138.65$,	$b = 226.19$,	$C = 59° \, 12' \, 54''$.
10.	$b = 1436.7$,	$c = 1141.2$,	$A = 42° \, 14' \, 35''$.

55. Case III. Given two sides and the angle opposite to one of them.

Geometric discussion. Let A, a, b be the given parts. We construct the angle XAY equal to the given angle A, and lay off AC, on AY, equal to the given side b. With C as center, and a radius equal to the given side a, we strike an arc. If this arc intersects AX in B and B', one or both of the triangles ABC and $AB'C$ *may* be solutions of the problem.

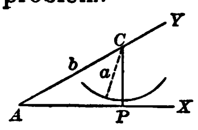

FIG. 48

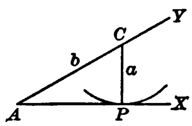

FIG. 49

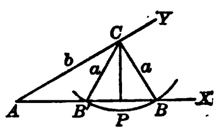

FIG. 50

The following cases may present themselves if A is an *acute* angle:

I. $a < CP$, i.e. $a < b \sin A$. (Fig. 48.)

The triangle is impossible in this case, since the arc of radius a, with C as center, will not intersect AX.

II. $a = CP = b \sin A$. (Fig. 49.)

The solution is the right triangle APC.

III. $b > a > b \sin A$. (Fig. 50.)

There are two solutions in this case. The triangles ABC and $AB'C$ *both* satisfy all of the requirements of the problem.

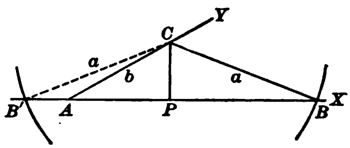

FIG. 51

IV. $a \geqq b$. (Fig. 51.)

If $a > b$, there is one solution only; namely, ABC. The triangle $AB'C$ does not contain the given angle A, but its supplement. If $a = b$, ABC is isosceles, and $AB'C$ reduces to a triangle of zero area. Thus we may say that there is one solution only if $a \geqq b$.

The following additional cases may present themselves if A is an *obtuse* angle.

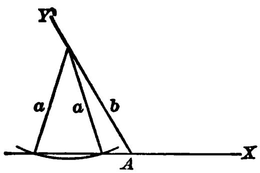

FIG. 52

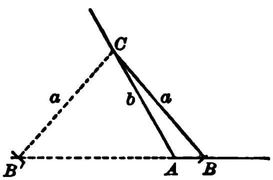

FIG. 53

V. $a \leqq b$. (Fig. 52.)

The triangle is impossible if $a < b$ and of zero area if $a = b$.

VI. $a > b$. (Fig. 53.)

There is one solution; namely, the triangle ABC. The triangle $AB'C$ does not contain the given angle A, but its supplement.

Trigonometric discussion. How do the equations of trigonometry bring into evidence these various cases?

The law of sines gives

$$(1) \qquad \sin B = \frac{b \sin A}{a}.$$

If A, a, b are given, we may compute the right member of this equation. Suppose first that A is an *acute* angle.

If the right member of (1) is greater than unity, the triangle is impossible, since the sine of an angle is never greater than unity. This is Case I of the geometric discussion.

If $\dfrac{b \sin A}{a} = 1$, B must be a right angle. (Case II.)

If the given values of a, b, and the acute angle A, are such that

$$(2) \qquad \frac{b \sin A}{a} < 1,$$

we can find not merely *one* angle to satisfy equation (1) but *two* such angles, which are supplementary to each other, one acute and one obtuse. For if K is the acute angle which satisfies equation (1), the obtuse angle $180° - K$, which has the same sine as K (See Art. 40), will also satisfy equation (1).

Thus we find in the first place, from (1), the two possibilities

$$(3) \quad B = K \text{ (acute angle)}, \quad B = 180° - K \text{ (obtuse angle)}.$$

We must remember, however, that $A + B$ must be less than 180°. If, as we have supposed, A is an acute angle, $A + K$ is certainly less than 180°. But $A + 180° - K$ is less than 180° if and only if $A < K$; that is (see Fig. 50, where $\angle K = \angle ABC$), if and only if $a < b$. Only in this case then will there be two solutions. That is, we have two solutions if A is acute and if $b > a > b \sin A$, in accordance with Case III of the geometric discussion.

If A is acute, but if $A + 180° - K \geqq 180°$; that is, if $A \geqq K$ and therefore $a \geqq b$, the obtuse angle solution $180° - K$ for B becomes inadmissible and the problem has only one solution, in agreement with Case IV of the geometric discussion.

If A is an obtuse angle, the obtuse angle solution $180° - K$ for B is never admissible, since a triangle can contain at

most one obtuse angle. The acute angle solution $B = K$ is admissible if and only if $A + K$ is less than 180°. This distinction gives rise to the two remaining Cases, V and VI. of the geometric discussion.

In the trigonometric solution of a numerical problem of this kind, it is essential to remember the following facts:

1. *If, on computing the sine of an angle, we find its value to be greater than unity, the triangle is impossible.*

2. *If the sine of an angle is found to be a positive proper fraction, there are two possibilities for the corresponding angle. One of these angles is acute and the other, the supplementary angle, is obtuse.*

3. *The sum of any two angles of a triangle must be less than* 180°.

The sine of B having been found from the law of sines, as indicated above, it will become apparent from the corresponding values of B whether the number of solutions is 0, 1, or 2. If there is one solution, we find C from

$$A + B + C = 180°$$

and c from the law of sines. We may check by one of Mollweide's equations or by the law of tangents. If both values of B are admissible, we use each of them in succession, so as to find the remaining parts of the two triangles which are solutions of the problem.

EXERCISE XXVI

EXAMPLE 1. Given $A = 15° \, 32'.7$, $a = 103.21$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

Formulæ: $\sin B = \dfrac{b \sin A}{a}$, $C = 180° - (A + B)$, $c = \dfrac{a \sin C}{\sin A}$.

Check: $b - a = \dfrac{c \sin \frac{1}{2}(B - A)}{\cos \frac{1}{2} C}$.

Given $\begin{cases} A = 15° \, 32'.7 & (1) \\ a = 103.21 & (2) \\ b = 152.37 & (3) \end{cases}$ Results $\begin{cases} B = 23° \, 18'.4 & B' = 156° \, 41.6 \\ C = 141° \, 8'.9 & C' = 7° \, 45.7 \\ c = 241.58 & c' = 52.01 \end{cases}$

Computation.

$\log b = 2.18290$	(4)
$\log \sin A = 9.42813$	(7)
$\operatorname{colog} a = 7.98628\text{–}10$	(6)
$\log \sin B = 9.59731\text{–}10$	(8)
$B = 23^\circ\, 18'.4$	(9)
$B' = 156^\circ\, 41'.6$	(10)
$B + A = 38^\circ\, 51'.1$	(11)
$B' + A = 172^\circ\, 14'.3$	(12)
$C = 141^\circ\ 8'.9$	(13)
$C' = 7^\circ\, 45'.7$	(14)

$\log a = 2.01372$	(5)
$\log \sin C = 9.79748\text{–}10$	(15)
$\operatorname{colog} \sin A = 0.57187$	(17)*
$\log c = 2.38307$	(18)
$c = 241.58$	(20)
$\log a = 2.01372$	(5)
$\log \sin C' = 9.13050\text{–}10$	(16)
$\operatorname{colog} \sin A = 0.57187$	(17)*
$\log c' = 1.71609$	(19)
$c' = 52.01$	(21)

Check.

$B - A = 7^\circ\, 45'.7$	(22)
$B' - A = 141^\circ\ 8'.9$	(23)
$\frac{1}{2}(B - A) = 3^\circ\, 52'.9$	(24)
$\frac{1}{2}(B' - A) = 70^\circ\, 34'.5$	(25)

$\log c = 2.38307$	(18)
$\log \sin \frac{1}{2}(B - A) = 8.83056\text{–}10$	(26)
$\operatorname{colog} \cos \frac{1}{2} C = 0.47811$	(28)
$\log (b - a) = 1.69174$	(30)
$b - a = 49.17$	(32)†
$b - a = 49.16$	(34)‡

$\log c' = 1.71609$	(19)
$\log \sin \frac{1}{2}(B' - A) = 9.97455\text{–}10$	(27)
$\operatorname{colog} \cos \frac{1}{2} C' = 0.00100$	(29)
$\log (b - a) = 1.69164$	(31)
$b - a = 49.16$	(33)‡

EXAMPLE 2. Given $A = 15^\circ\, 32'.7$, $a = 10.321$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

$$\text{Given} \begin{cases} A = 15^\circ\, 32'.7 \\ a = 10.321 \\ b = 152.37 \end{cases}$$

$$\begin{aligned} \log b &= 2.18290 \\ \log \sin A &= 9.42813\text{–}10 \\ \operatorname{colog} a &= 8.98628\text{–}10 \\ \hline \log \sin B &= 0.59731 \end{aligned}$$

Since $\log \sin B$ has the characteristic zero, $\sin B$ is greater than unity. Therefore, the triangle is impossible.

EXAMPLE 3. Given $A = 15^\circ\, 32'.7$, $a = 167.38$, $b = 152.37$. Find the remaining parts of the triangle or triangles determined by these data.

Solution.

$$\text{Given} \begin{cases} A = 15^\circ\, 32'.7 \\ a = 167.38 \\ b = 152.37 \end{cases} \qquad \text{Results} \begin{cases} B = 14^\circ\ 7'.2 \\ C = 150^\circ\, 20'.1 \\ c = 309.11 \end{cases}$$

---

* Obtained from (7).

† Obtained from the logarithm above.

‡ Obtained by subtraction from (2) and (3).

Computation.

$$\begin{aligned}
\log b &= 2.18290 \\
\log \sin A &= 9.42813\text{–}10 \\
\operatorname{colog} a &= 7.77629\text{–}10 \\
\hline
\log \sin B &= 9.38732\text{–}10 \\
B &= 14^\circ\ 7'.2 \\
B' &= 165^\circ\ 52'.8 \\
B + A &= 29^\circ\ 39'.9 \\
B' + A &= 181^\circ\ 25'.5^* \\
C &= 150^\circ\ 20'.1
\end{aligned}
\qquad
\begin{aligned}
\log a &= 2.22371 \\
\log \sin C &= 9.69454\text{–}10 \\
\operatorname{colog} \sin A &= 0.57187 \\
\hline
\log c &= 2.49012 \\
c &= 309.11
\end{aligned}$$

Check.

$$\begin{aligned}
A - B &= 1^\circ\ 25'.5 \\
\tfrac{1}{2}(A-B) &= 0^\circ\ 42'.8 \\
\tfrac{1}{2} C &= 75^\circ\ 10'.1 \\
a - b &= 15.01
\end{aligned}
\qquad
\begin{aligned}
\log c &= 2.49012 \\
\log \sin \tfrac{1}{2}(A - B) &= 8.09516\text{–}10 \\
\operatorname{colog} \cos \tfrac{1}{2} C &= 0.59180 \\
\hline
\log (a - b) &= 1.17708 \\
a - b &= 15.03
\end{aligned}$$

Find out whether the triangles corresponding to the following data are possible, how many solutions there are, and what are the values of the missing parts.

4. $a = 98$, $b = 100$, $A = 120^\circ$.
5. $a = 767$, $b = 242$, $A = 36^\circ\ 53'\ 2''$.
6. $a = 3541$, $b = 4017$, $A = 61^\circ\ 27'$.
7. $a = 67.53$, $b = 56.82$, $A = 77^\circ\ 14'\ 19''$.
8. $a = 9.4672$, $c = 14.433$, $A = 11^\circ\ 14'.3$.
9. $a = 413.28$, $b = 378.19$, $B = 50^\circ\ 16'\ 25''$.
10. $a = 345.46$, $b = 531.75$, $A = 26^\circ\ 47'\ 32''$.

56. Case IV. Given the three sides of the triangle. We use the formulæ

$$s = \tfrac{1}{2}(a + b + c), \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

$$\tan \tfrac{1}{2} A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2} B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2} C = \frac{r}{s-c},$$

and the check

$$A + B + C = 180^\circ.$$

---

* Therefore B' is inadmissible. There is only one solution, as might have been foreseen, since $a > b$.

EXERCISE XXVII

Example 1. Given $a = 34.278$, $b = 25.691$, $c = 30.175$. Find the angles A, B, C.

Solution.

$$\text{Given}\begin{cases} a=34.278\\ b=25.691\\ c=30.175\end{cases} \qquad\qquad \text{Results}\begin{cases} A=\ 75^\circ\ 13'.0\\ B=\ 46^\circ\ 26'.6\\ C=\ 58^\circ\ 20'.2\end{cases}$$

$$\begin{array}{r} 2s=90.144\\ \hline s=45.072\\ s-a=10.794\\ s-b=19.381\\ s-c=14.897\\ \hline 2s=90.144^{*}\end{array} \qquad\qquad \textit{Check.}\ \begin{array}{r} A+B+C=179^\circ\ 59'.8\\ \hline \operatorname{colog} s=8.34609-10\\ \log(s-a)=1.03318\\ \log(s-b)=1.28737\\ \log(s-c)=1.17310\\ \hline \log r^2=1.83974\dagger\end{array}$$

$\log r=0.91987$	$\log r=0.91987$	$\log r=0.91987\ddagger$
$\log(s-a)=1.03318$	$\log(s-b)=1.28737$	$\log(s-c)=1.17310$
$\log\tan\frac{1}{2}A=9.88669-10$	$\log\tan\frac{1}{2}B=9.63250-10$	$\log\tan\frac{1}{2}C=9.74677-10$
$\frac{1}{2}A=37^\circ\ 36'.5$	$\frac{1}{2}B=23^\circ\ 13'.3$	$\frac{1}{2}C=29^\circ\ 10'.1$

Remark. In this problem some time may be saved by writing $\log r$ on the lower margin of a slip of paper and placing it above $\log(s-a)$ to find $\log\tan\frac{1}{2}A$, above $\log(s-b)$ to find $\log\tan\frac{1}{2}B$, and above $\log(s-c)$ to find $\log\tan\frac{1}{2}C$. A similar device is often useful in similar cases. Most computers also save time by omitting the -10 attached to logarithms with negative characteristics. This omission can never give rise to serious misunderstanding.

Find the angles of the triangles whose sides have the following values:

2. $a = 79.3$, $b = 94.2$, $c = 66.9$.
3. $a = 0.785$, $b = 0.850$, $c = 0.633$.
4. $a = 312$, $b = 423$, $c = 342$.
5. $a = 25.17$, $b = 34.06$, $c = 22.17$.
6. $a = 93146$, $b = 176530$, $c = 95768$.
7. $a = 12.653$, $b = 17.213$, $c = 23.106$.

If only *one* of the three angles is to be calculated, it may be more convenient to make use of the formulæ for $\sin\frac{1}{2}A$ or $\cos\frac{1}{2}A$, which may be found from the indications given in Example 4, Exercise XXII.

---

* Obtained by adding s, $s-a$, $s-b$, $s-c$, as a check on the additions and subtractions required to find these quantities.

† Obtained by adding the four logarithms above $\log r^2$, since

$$r^2=\frac{(s-a)(s-b)(s-c)}{s}.$$

‡ Obtained by taking one half of $\log r^2$.

57. Problems in heights and distances, plane surveying, and plane sailing. Some of the following problems are direct applications of the methods which have just been explained. In others, it will be necessary to consider several triangles in succession or simultaneously.

It will always be advisable to draw a figure, approximately to scale, to denote the known as well as the unknown sides and angles of the figure by properly chosen letters, and to write down the formulæ to be used in their general form, leaving the substitution of the numerical values to the last. Much blundering and much unnecessary work may be avoided by adopting this plan.

The student should use his judgment in regard to the number of decimal places used in the numerical part of the work. Use three-place tables whenever possible. Many of the problems may be solved, wholly or in part, by the slide-rule.

EXERCISE XXVIII

1. In order to find the height of a tower, the angles of elevation of its top are measured from two stations, A and B, in the same horizontal line with its base, and on the same side of the tower. If the angles of elevation of the tower from A and B are 32° and 65° respectively, and if the distance AB is 500 feet, find the height of the tower.

2. Find the distances from the two stations of Ex. 1 to the foot of the tower.

3. If the angles of elevation of the tower from A and B, in Ex. 1, are L and M respectively, and if the distance AB is equal to d feet, show that the height of the tower is

$$h = \frac{d \sin L \sin M}{\sin (M - L)}.$$

4. An obstacle (a house) was found to interfere with the running of a straight line from A in the direction AB. (See Fig. 54.) An angle ABE was turned at B, equal to 123°, and the distance BE was measured equal to 150 feet. The angle BEC was made equal to 63°. How long must the distance EC be, and what angle must be turned at C, in order that CD may be the prolongation of AB?

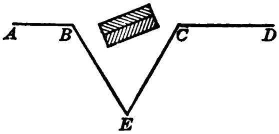

FIG. 54

5. How may angles at B and E be chosen, in Ex. 4, so as to avoid all computation?

6. Two straight railroad tracks intersect at an angle of 75°. What will be the distance, at the end of 20 minutes, between two trains which start from the crossing at the same instant, their speeds being 30 and 40 miles per hour respectively?

7. Each of two battleships passing each other fired a salute. A person on shore observed that the interval between the flash and the report of the gun was 4 seconds for one ship and 6 seconds for the other. The angle, at his eye, subtended by the two ships, just as the salute was fired, was 55°. The velocity of sound is about 1140 feet per second. Find the distance between the ships.

8. Two lighthouses are 2.789 miles apart, and a certain rock is known to be 4.325 miles from one of them. The angle subtended by the two lighthouses at the rock is 16° 13′. How far is the rock from the other lighthouse? How many solutions are there to this problem? Can we make a choice between these solutions if we know which of the two lighthouses is nearer to the rock?

9. Find the radius of the largest cylindrical gas tank which can be constructed on a triangular lot whose sides measure 73, 82, 91 feet respectively, and locate the center of its circular base.

10. An observer measures the angle of elevation of a cloud due south of him at the moment when the sun also is due south (at apparent noon). The angle of elevation of the sun was 65°, that of the cloud 75°. If the shadow of the cloud falls 550 feet north of the observer, how high is the cloud?

11. At 9 P.M. two lights, known to be 8 miles apart, are observed to be due east from a certain vessel. At 10 P.M. one of these lights bears N.E. and the other N.N.E. If the course of the ship was due south, what was its rate?

12. A tower is situated on top of a conical hill whose sides make an angle of 15° with the horizontal plane. At a distance of 120 feet from the foot of the tower (the distance being measured along the slope) the tower subtends an angle of 20°. Find the height of the tower.

13. If, in Ex. 12, the side of the hill makes an angle I with the horizontal plane, and if the angle subtended by the tower, at a distance of d feet from its foot, is A, show that the height of the tower is

$$h = \frac{d \sin A}{\cos (A + I)}.$$

14. A tower 54 feet high, situated on top of a conical hill, subtends an angle of 15° 30′ at a point 120 feet from the foot of the tower (the distance being measured along the slope). What angle does the side of the hill make with the horizontal plane?

15. To find the slope of a railroad embankment, one end of a pole 12 feet long was placed on the level ground 6 feet from the foot of the embankment, and the other end was found to fall at a point 7.5 feet up its face. What angle does the embankment make with a horizontal plane?

Remark. A transit cannot conveniently be used to measure an angle formed by two walls, the angle formed by an embankment or buttress with a horizontal plane, etc. In such cases, as in this example, it is more convenient to measure distances and determine the angles by calculation.

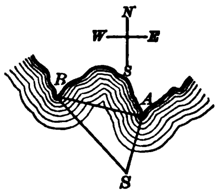

Fig. 55

16. Two capes, A and B (Fig. 55), were observed from a ship at sea; one of them bore N.N.E. and the other N.W. It was found from the chart that the second cape bore W. by N. from the first and was 25.3 miles distant from it. What was the distance of the ship from each of the two capes?

17. A battleship leaves port A, on a due easterly course, at the rate of 16 miles per hour. A dispatch boat starts from B at the same moment. The port B bears S.S.W. of port A and is 25 miles distant from it. If the dispatch boat has a rate of 22 miles per hour, what should be the direction of its course so that it may meet the battleship, if neither ship alters its rate or course? At what time will they meet?

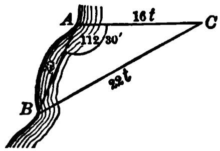

Fig. 56

Hint. In Fig. 56 we have $AC = 16\,t$, $BC = 22\,t$, if t denotes the time (in hours) which passes between the time of sailing and the moment of meeting, and if C represents the place of meeting.

18. The angle of elevation of the top of a tower, at a point in the same horizontal plane with its base, is equal to A. At a point h feet directly above the first the angle of depression of the foot of the tower was found to be equal to B. Prove that the height of the tower is equal to $h \tan A \cot B$.

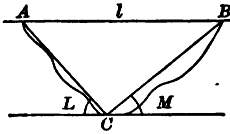

Fig. 57

19. A valley has the cross section shown in Fig. 57, the angles L and M and the distance $AB = l$ having been obtained by a survey. It is planned to connect the points A and B by a

bridge, supported by a pier at C. How high must this pier be made?

20. Show that the area of a quadrilateral is $\frac{1}{2} dd' \sin A$, if d and d' are the lengths of its diagonals and if A is one of the angles which the diagonals make with each other.

21. Two sides of a parallelogram are 3.41 and 2.60 feet long and the shorter diagonal is 1.58 feet long. Find the length of the other diagonal.

22. The sides of a field $ABCD$ are: $AB=57$ feet, $BC=43$ feet, $CD=45$ feet, $DA=47$ feet; and the distance from A to C is 50 feet. Find the area of the field.

23. Two streets intersect at an angle of 75°. The corner lot has frontages of 150 feet and 115 feet on the two streets, and the remaining two boundary lines of the lot are perpendicular to the two streets. What is their length, and what is the area of the lot?

24. In order to measure the distance between two pumping stations, A and B, in Lake Michigan, a base line $CD = 17.7$ chains was measured along the shore. (See Fig. 58.) The following angles were measured:

$$ACD = C_1 = 132° \, 29',$$
$$ACB = 82° \, 20',$$
$$CDA = D_1 = 45° \, 59',$$
$$CDB = D_2 = 124° \, 48'$$

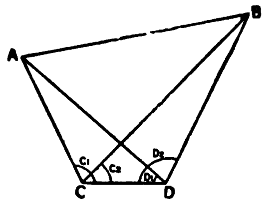

FIG. 58

Compute the distance AB.

(Fig. 58 is not drawn to scale.)

25. Devise a plan for finding the distance between any two inaccessible points, A and B, in the same horizontal plane if two points, C and D, can be found in the same plane, from both of which A and B are visible.

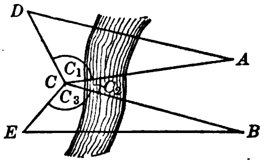

FIG. 59

26. Devise a plan for finding the distance between two inaccessible points, A and B, if both are visible from only *one* accessible point C.

HINT. In Fig. 59, select a point D from which A and C are visible, and a point E from which B and C are visible. Measure CD, CE, and the angles D, C_1, C_2, C_3, E.

27. To compute the distance between two accessible points, A and B, if no point can be found from which both A and B can be seen. (For instance, if A and B are points on opposite sides of an inaccessible mountain.) Take a point C from which A may be seen and a point D from which B is visible. If C is visible from D, measure the angles ACD and CDB and the distances AC, CD, DB. Show how to calculate the distance AB from these data.

28. If the points A and B of Ex. 27 are inaccessible, the distances AC and BD cannot be found by direct measurement. In such a case (see Fig. 60), select points C, D, E, F so that A, C, E shall be visible from D, and D, F, B from E. Measure the angles C, D_1, D_2, E_1, E_2, F, and the distances CD, DE, and EF. Show how to find the distance AB from these measurements.

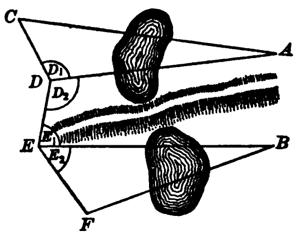

FIG. 60

29. A tower is situated on top of a conical hill as in Ex. 12. Two points A and A' are chosen on the side of the hill at distances $d = 43$ feet and $d' = 98$ feet respectively from the top, the point A' being the lower one and the distances d and d' being measured along the slope. The angles subtended by the tower at A and A' were $A = 42°$, $A' = 23°$ respectively. Find the height of the tower and the angle of inclination of the side of the hill.

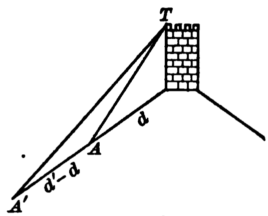

FIG. 61

30. Two observers, A and B, are 3 miles apart, A being due west of B and in the same horizontal plane. Both observers measure, at the same instant, the bearing and angle of elevation of a balloon. A finds that the balloon bears N. 47° E. and that its angle of elevation is 23°. B finds that the balloon bears N. 35° W. Find the height of the balloon above the horizontal plane of A and B and the angle of elevation at B.

31. To find the horizontal distance AD and the vertical distance DC from A to an inaccessible point C (Fig. 62) when it is not convenient to measure a base line in the same vertical plane with C.

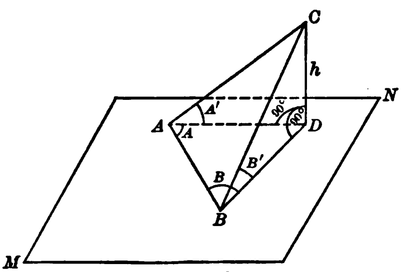

FIG. 62

Measure a horizontal base line AB, k feet long, in any direction through A. Let D be the foot of the perpendicular from C to the horizontal plane, MN, of AB. Measure the horizontal angles $BAD = A$ and $ABD = B$, and the vertical angles $DAC = A'$ and $DBC = B'$.

Show that
$$AD = \frac{k \sin B}{\sin (A + B)}, \quad BD = \frac{k \sin A}{\sin (A + B)},$$
$$h = \frac{k \sin B \tan A'}{\sin (A + B)} = \frac{k \sin A \tan B'}{\sin (A + B)}.$$

The two formulæ for h should give the same result in any numerical problem of this kind. Lack of agreement indicates inaccuracy in one of the observed angles or in the computation.

32. Two points, A and B, in the same horizontal plane and separated by a ridge, are to be connected by a straight level tunnel. In order to find the distance between them, the surveyors measured the inclined angle, BCA, subtended by AB from the top C of a neighboring hill, whose height, $CD = h$, above the plane of A and B is known. They also measured the angles of depression of A and B from C. Devise a method for computing the length of the tunnel AB.

33. Prove, by means of the trigonometric formulæ, that the angles of a triangle can be found when the *ratios* of the three sides of the triangle are given, even if the absolute values of the three sides are not known. Prove the same fact by geometry.

34. Let A, B, C be three points of a horizontal line, whose mutual distances, $AB = b$, $AC = c$, $BC = b + c$, are known. (See Fig. 63.) To find the horizontal distances, p, q, r, of an inaccessible point, E, from these three points, and the height h, it suffices to measure the three angles of elevation, A, B, C, of E from A, B, C respectively.

For the figure gives

(1) $p = h \cot A, \qquad q = h \cot B,$
$r = h \cot C,$

and also

$$q^2 = b^2 + p^2 - 2\,bp \cos BAD,$$
$$r^2 = c^2 + p^2 + 2\,cp \cos BAD;$$

whence

$$\frac{b^2 + p^2 - q^2}{b} = \frac{r^2 - c^2 - p^2}{c}.$$

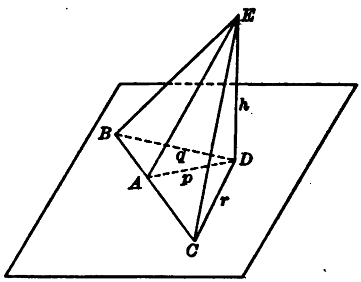

FIG. 63

Substitute the values of p, q, r from (1) and solve the resulting equation for h^2. We find

$$(2) \qquad h^2 = \frac{bc(b + c)}{(\cot^2 C - \cot^2 A)b + (\cot^2 B - \cot^2 A)c}.$$

After computing h from (2), p, q, and r may be found from (1).

35. Given the mutual distances of three points A, B, C, to find the distances AD, BD, CD from a fourth point D in the plane of ABC to each of the given three points, when the angles are given which the lines AB, BC, CA subtend at D.

This problem, usually called *Pothenot's* problem, may be solved as follows:

In Fig. 64, let

$$BC = a, \quad CA = b, \quad AB = c$$

be the given mutual distances of A, B, and C. Of course the angles of the triangle ABC may then also be regarded as known.

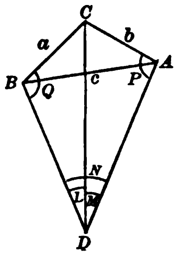

FIG. 64

Let $\angle CDB = L, \quad \angle CDA = M, \quad \angle BDA = N$

be the angles subtended by the sides of the triangle from D. These angles we regard as known by measurement and, of course, $N = M + L$.

The problem of finding the distances AD, BD, CD may evidently be regarded as solved, if we can find the two angles

$$\angle CAD = P \text{ and } \angle CBD = Q.$$

We shall show how to compute these angles.

Applying the law of sines to the triangles ACD and BCD, we find

$$(1) \qquad CD = \frac{b \sin P}{\sin M} = \frac{a \sin Q}{\sin L},$$

whence

$$\frac{\sin P}{\sin Q} = \frac{a \sin M}{b \sin L}.$$

Hence, by the theory of proportion,

$$\frac{\sin P - \sin Q}{\sin P + \sin Q} = \frac{a \sin M - b \sin L}{a \sin M + b \sin L},$$

which gives (Equations (8), Art. 49),

$$(2) \qquad \frac{\tan \frac{1}{2}(P - Q)}{\tan \frac{1}{2}(P + Q)} = \frac{1 - \dfrac{b \sin L}{a \sin M}}{1 + \dfrac{b \sin L}{a \sin M}}.$$

On the other hand we have

$$P + Q + C + N = 360°,$$

and therefore

$$(3) \qquad \tfrac{1}{2}(P + Q) = 180° - \tfrac{1}{2}(C + N).$$

Since the angles C and N are known, (3) gives the value of $\frac{1}{2}(P + Q)$. If this value be substituted in (2), we may obtain from (2) the value of $\frac{1}{2}(P - Q)$. From $\frac{1}{2}(P + Q)$ and $\frac{1}{2}(P - Q)$, we find P and Q themselves by addition and subtraction. The distance CD may then be computed by (1) in two ways, providing a check for the correctness of the work.

Remark 1. The problem obviously becomes indeterminate, if the point D should happen to be on the circumference of the circle determined by the three points A, B, C. For, as the point D moves along this circumference, the angles L and M do not change, so that the position of the point D on the circumference is not determined by the value

of these angles. It is evident, then, that if the point D is close to the circumference of the circle circumscribed about the triangle ABC, its position cannot be determined by this method with any considerable degree of accuracy.

Remark 2. The name POTHENOT's problem cannot be justified historically. A complete solution of this problem was given by SNELLIUS in his *Doctrinæ triangulorum canonicæ libri quatuor*, which appeared in 1627, almost seventy years before Pothenot presented his solution of the same problem to the Paris Academy of Science.

36. Show that the problem of Pothenot may also be solved by drawing a circle through the three points A, B, D, and by making use of the triangle ABE, where E is the second point of intersection of CD with this circle.

37. In surveying a harbor, a submerged rock was located, for charting purposes, by sighting three known objects A, B, C on land from a boat immediately above the rock. The known distances were $BC = a = 312$ feet, $CA = b = 520$ feet, and the angle C was known to be $65° 27'$. The angles obtained by observation were $L = CDB = 23° 25'$ and $M = CDA = 32° 52'$. Find and check the distance from the rock to C and the angle which this line makes with the side AC of the known triangle.

In Exs. 38 to 45, we use the notations of Chapter VII; A, B, C for the angles, a, b, c for the sides, $2s$ for the perimeter, r and R for the radii of the inscribed and circumscribed circle respectively, and S for the area of the triangle. Show how to find all of the sides and angles of the triangles determined by the following data.

38. $a + b$, c, $A - B$ are given.

39. $a - b$, c, $A - B$ are given.

40. R, a, b are given.

41. R, A, B are given.

42. r, A, B are given.

43. S, A, B are given.

44. S, a, b are given.

45. s, R, a are given.

46. A furnace maker receives an order for a number of furnaces, some 40 inches and some 42 inches in diameter. These furnaces are to be fitted on the outside with an iron casting whose inside length, measured along the arc, is 26 inches. In order to avoid the necessity of making two different castings, the manufacturer considers the possibility of making a single casting, to fit *exactly* a furnace 41 inches in diameter but, of course, not fitting exactly either of the sizes ordered. His experience tells him that such a casting will serve the purpose, if no point of its inner surface is more than a quarter of an inch from the outer surface of the furnace after being placed in position. Will it be necessary to make separate castings for the two different sizes?

58. Displacements, velocities, and forces. If a body is transported from one place in the plane to another, and we wish to describe its change of position or *displacement*, it is clearly not sufficient to state *how far* the body has been moved. We must also include in our description a statement concerning the *direction* of the displacement.

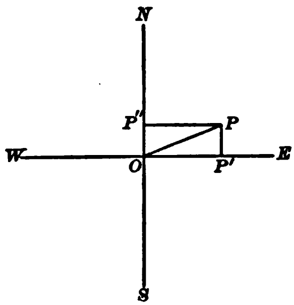

FIG. 65

Thus, the displacement from O to P (Fig. 65) may be described completely by stating its *magnitude* (the length of the line-segment OP), and its *direction* (either the bearing NOP of the line OP or the angle EOP or some other angle which fixes the direction of OP).

If the displacement is in a horizontal plane and the directions from O to N, S, E, W in Fig. 65 represent north, south, east, and west respectively, the projections OP' and OP'' of OP on OE and ON are called the easterly and northerly *components* of the displacement. If the displacement is not horizontal, we define, in a similar manner, its horizontal and vertical components.

Let us suppose that a point M is displaced from O to P. (See Fig. 66.) The displacement may be represented in magnitude and direction by the directed line-segment OP. (The arrowhead indicates that the displacement is from O toward P, and not from P toward O.) Let OQ represent a second displacement. If the point M, originally at O, be made to undergo both of these displacements in succession (in either order), it will ultimately arrive at R, where R is the fourth vertex of the parallelogram determined by OP and OQ. For this reason, the displacement OR is said to be the *resultant* of the displacements OP and OQ.

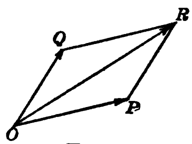

FIG. 66

We may think of the two displacements OP and OQ as taking place simultaneously. An instance of this sort is

furnished by a passenger shifting his position on board of a moving ship. His total displacement, in space, is the resultant of that which is due to the motion of the ship and of that caused by his own muscular efforts.

The *velocity* of a train, of a bullet, or of any uniformly moving object, is measured by the distance which it describes in a unit of time, that is, by a displacement. Therefore a velocity, like a displacement, has direction as well as magnitude. The case of a passenger's stroll on board a moving ship suffices to illustrate the phrase: *resultant of two velocities*. Since the velocity of a uniformly moving body is its displacement in a unit of time, the resultant of two velocities is found by the same method as a resultant of two displacements, *i.e.* by the parallelogram construction illustrated in Fig. 66.

It is one of the fundamental facts of mechanics (proved by countless experiments) that *two forces acting upon the same material point combine into a single* **resultant** *force according to this same parallelogram law*. This fact is generally known as the *law of the parallelogram of forces*.

Owing to the fact that displacements, velocities, and forces are directed quantities which combine according to the parallelogram law, these three classes of things have many properties in common. There are many other instances of quantities of this same character and, on account of their importance, they have received a special name.

Directed quantities, which combine in accordance with the parallelogram law, are called ***vectors***.*

By a proper choice of the units, every vector may be represented by a directed line-segment or, what amounts to the same thing, by a displacement.

Thus, the line-segments of Fig. 66 may be interpreted as forces, if the directions of these line-segments coincide with the directions of the forces, and if each segment is made to contain as many length units as there are force units in the corresponding force.

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* From the Latin *vector*, meaning one who carries or conveys.

EXERCISE XXIX

1. A steamer is moving N.N.E. with a velocity of 16 miles per hour. Find the northerly and easterly components of its velocity.

2. A horizontal force of 10 lb. and a vertical force of 24 lb. are acting simultaneously on a point. Find the magnitude and direction of the resultant.

3. A schooner is sailing due west at the rate of 6 miles per hour. A sailor is crossing the deck, from south to north, at the rate of 3 miles per hour. What is the magnitude and direction of his velocity in space?

4. A force of 250 lb. is acting on a body in a direction which makes an angle of 17° with the horizontal plane. How much of this force tends to lift the body, and what part of it tends to move the body in a horizontal plane?

5. Two forces, of magnitudes 350 lb. and 510 lb., respectively, act upon the same point, in directions which make an angle of 35° with each other. Find the magnitude of the resultant, and the angles which it makes with each of the component forces.

6. A force of 216 lb. is resolved into two components, which make angles of 27° and 32° respectively, with the direction of the original force. Find the magnitude of each component.

7. A man wishes to reach a point on the opposite side of the river, 250 yards upstream. The velocity of the current is 2.5 miles per hour and the width of the river is 300 yards. If the man's rate of rowing (in still water) is 4 miles per hour, in what direction must he point the head of the boat in order that his course may be a straight line?

59. Reflection and refraction of light. The path of light, in a homogeneous medium like air, is rectilinear. But if a ray of light meets the polished surface of a sheet of metal or glass, its direction is changed in accordance with the law that *the angle of incidence is equal to the angle of reflection.*

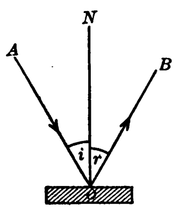

FIG. 67

This law is illustrated in Fig. 67, where the ray AO strikes the reflecting surface of the mirror at O and is reflected in the direction OB. The line ON, perpendicular to the reflecting surface at O, is called its normal. The angle NOA, or i, is called the *angle of incidence*, and the angle NOB, or r, is the *angle of reflection.* According to the law of reflection of light (veri-

fied by thousands of experiments) these two angles are equal.

When a ray of light AO, after passing through air, meets the bounding surface of a second transparent medium like glass (see Fig. 68), only a part of the light is reflected. Another portion of the light enters the second medium and continues on its way in a path OB, which makes a certain angle with the direction AO of the original ray.

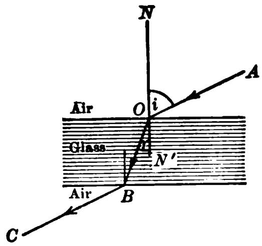

FIG. 68

If NN' is the perpendicular or normal to the bounding surface at O, the angle NOA, or i, is called the *angle of incidence* and $N'OB$, or r, is the *angle of refraction.*

As a result of numerous experiments, it has been found that the quotient

$$\frac{\sin i}{\sin r}$$

will have the same value, for a given kind of glass, for all different values of i. In other words, as the angle of incidence changes the angle of refraction also changes, but in such a way as to leave the quotient

$$\frac{\sin i}{\sin r} = n \tag{1}$$

unchanged. This quotient n is called the **index of refraction** of the glass with respect to air. Its value is different for different kinds of glass. For ordinary crown glass n is about equal to 1.5.

If the ray of light again emerges into air, as indicated in Fig. 68, after having passed through a sheet of glass with exactly parallel sides, careful measurements show that the ray BC is parallel to the original ray AO. In other words, the direction of a ray of light is not changed by passing through a sheet of plate glass whose two faces are exactly parallel to each other. Therefore: *the index of refraction of*

air with respect to glass is the reciprocal of the index of refraction of glass with respect to air.

The law of the refraction of light was first discovered by SNELLIUS in 1618. The simple formula (1) was given by DESCARTES in 1637.

EXERCISE XXX

1. A light is placed on the line perpendicular to a plane circular mirror at its middle point. The distance from the light to the mirror is 15.76 inches, and the mirror is 8.32 inches in diameter. Consider two rays which strike the mirror in the extremities of one of its diameters. What angle will they make with each other after reflection?

2. In an experiment, a source of light is to be placed at A, a mirror at B, and a photographic plate at C. The distances are: $AB = 5.367$ meters, $BC = 6.329$ meters, $CA = 7.361$ meters. What angle must the mirror at B make with the line AB so that the light, reflected at B, may pass through C?

3. Two billiard balls, A and B, have been placed at distances a and b inches respectively from the same cushion. The line joining them makes an angle L with the cushion. Let K be the angle at which the first ball must strike the cushion, so as to hit the second after rebounding. Show that

$$\tan K = \frac{b+a}{b-a} \tan L.$$

Remark. Billiard balls not endowed with a lateral rotation (without "English") rebound in accordance with the law of reflected light.

4. Find the index of refraction from the following observations. (Wüllner)

Angles of incidence, i . . .	40°	60°	80°
Angles of refraction, r . . .	24° 24′	33° 38′	38° 57′

The three values obtained for n will disagree slightly owing to inaccuracies in the measurements.

5. A ray of light strikes a plate of crown glass at an angle of incidence of 37°. Find the angle between the reflected and the refracted ray, if the index of refraction is 1.559.

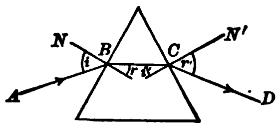

FIG. 69

6. A ray of light $ABCD$ passes through a glass prism whose cross section (see Fig. 69) is an equilateral triangle. If the index of refraction is 1.559, what must be the angle of incidence in order that the path of the light in the prism may be parallel to one of its faces? What angle will the ray CD make with its original direction AB, after emerging from the prism?

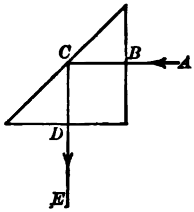

FIG. 70

7. A ray of light, ABC, etc., enters a glass prism, whose cross section is an isosceles right triangle and whose index of refraction is 1.5, at the point B (Fig. 70), at right angles to the face of the prism. At C no part of the ray can be refracted (Why?), and all of it is reflected in the direction CE. Such a prism is called a total reflecting prism.

THE GREEK ALPHABET

α,	A	Alpha	ν,	N	Nu
β,	B	Beta	ξ,	Ξ	Xi
γ,	Γ	Gamma	o,	O	Omicron
δ,	Δ	Delta	π,	Π	Pi
ϵ,	E	Epsilon	ρ,	P	Rho
ζ,	Z	Zeta	σ, ς,	Σ	Sigma
η,	H	Eta	τ,	T	Tau
θ, ϑ,	Θ	Theta	υ,	Υ	Upsilon
ι,	I	Iota	ϕ,	Φ	Phi
κ,	K	Kappa	χ,	X	Chi
λ,	Λ	Lambda	ψ,	Ψ	Psi
μ,	M	Mu	ω,	Ω	Omega

PART TWO

PROPERTIES OF THE TRIGONOMETRIC FUNCTIONS

CHAPTER IX

THE GENERAL ANGLE AND ITS TRIGONOMETRIC FUNCTIONS

60. The notion of the general angle. In elementary geometry we usually think of an angle as ready-made. We there think of two lines as given and understand by the angle between them a measure of their difference of direction. But many reasons urge us to oppose to this *static* idea what might be called the *dynamic* concept of angle, which presents an angle, not as the ready-made difference of direction between two fixed lines, but as something which is *generated* by the rotation of a straight line around a fixed point as pivot.

Thus, for instance, we shall say that the minute hand of a clock describes, or generates, an angle of 90° in fifteen minutes, an angle of 360° in one hour, an angle of 1890° in five hours and a quarter. Although the minute hand points to the same place on the face of the clock after any number of complete revolutions, we are not likely to make the error of ignoring these complete revolutions.* If we did, we should be ignoring the distinction between 1 o'clock, 2 o'clock, 3 o'clock, etc. If we were to say that an angle of 360° is the same as one of 0°, or that an angle of 450° is equal to one of 90°, we should be committing the same error.

We see that, while an angle in the sense of elementary geometry can never be greater than 180°, our new concept of angle permits us to speak of angles of *any* magnitude.

* It is the purpose of the hour hand to record the number of complete revolutions.

Our notions will be enriched in still another way, if we adopt the dynamic instead of the static concept of angle. The line, whose rotation generates the angle, may revolve in either of two opposite directions, clockwise or counterclockwise; and we must distinguish between these two kinds of rotation, just as we distinguish between two motions in opposite directions on a straight line. This distinction may be made by ascribing to every angle, not merely a *magnitude*, but also a *sign* depending upon the direction of the rotation by which the angle is generated.

It is customary to speak of counterclockwise rotations as positive, and of clockwise rotations as negative.

The reason for this convention * will appear later (Art. 63).

EXERCISE XXXI

Using a protractor, combine the following angles graphically and check the results arithmetically.

1. $30° + 60°$, $50° - 30°$, $30° - 60°$, $50° + (-30°)$, $30° + (-60°)$.

2. $25° + 15° - 35°$, $135° - (-25°) + 150°$.

3. $225° + 345° - 185°$, $30° + (3 \times 15°)$.

4. What angle does the minute hand of a clock describe in 3 hours and 25 minutes? in 5 hours and 13 minutes?

5. Suppose that the dial of a clock is transparent so that it may be read from both sides. Each of two persons, stationed on opposite sides of the dial, observes the motion of the minute hand for fifteen minutes. Upon comparing notes, they find that they do not agree in regard to the angle described by the minute hand during this period of time. In what respect do they differ?

6. What is the magnitude of the angle described by a spoke of a carriage wheel, 3 feet in diameter, when the carriage travels a distance of 500 feet?

Note. Think of the wheel as if it were turning on the axle while the carriage is standing still.

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* The word *convention* is here used in a special sense, meaning an *arbitrary agreement.*

7. The earth describes an approximately circular orbit about the sun as center in 365 days. What angle will the line joining the sun to the earth (the earth's *radius vector*) describe in 415 days?

8. Two wheels, A and B, are joined by a belt as in Fig. 71. The diameter of A is twice that of B, and A is moving in counterclockwise direction. What angle will a spoke of B describe while A rotates through an angle of 300°?

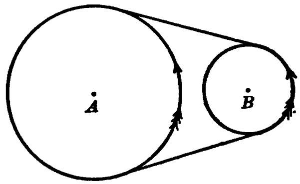

FIG. 71

9. If the two wheels of Ex. 8 are joined by a crossed belt, what angle will a spoke of B describe when A rotates through an angle of 300°?

10. If n wheels are connected by gears, what kind of a number must n be in order that the first and last wheel may rotate in the same direction? in opposite directions?

61. Initial and terminal side. Standard position of an angle. Our new concept of an angle, as a measure for the amount of rotation of a line, leads us to distinguish between the **initial** and **terminal** sides of an angle.

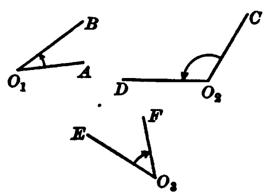

FIG. 72

Thus, in Fig. 72, we have three angles whose senses of rotation are indicated by curved arrows. The angles AO_1B and CO_2D are positive, for the rotation is counterclockwise. Angle EO_3F is negative. The *initial* sides of these angles are AO_1, CO_2, EO_3, and their *terminal* sides are BO_1, DO_2, and FO_3 respectively.

If an angle is thought of as generated by the rotation of a straight line, the initial and final positions of this line are called the initial and terminal sides of the angle respectively.

If we wish to compare two parallel directed line-segments in regard to magnitude and sign, we usually think of one of them as being moved, until its initial point coincides with the initial point of the other. In the same way, in order to compare two angles we usually place them so that their vertices and initial sides shall coincide.

It is customary, for purposes of comparison, to place all

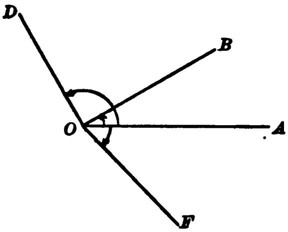

FIG. 73

angles in such a position that their initial sides are on a horizontal line and pointed toward the right. An angle placed in this way shall be said to be in its **standard position.**

Thus, Fig. 73 represents the three angles of Fig. 72 in their standard positions; the angles AOB, AOD, and AOF of Fig. 73 being equal to the three angles AO_1B, CO_2D, and EO_3F of Fig. 72 respectively.

EXERCISE XXXII

Place the following angles in their standard positions:

1. 15°, 225°, 415°, 768°.

2. – 25°, – 275°, – 615°, – 365°.

3. If two angles differ by an integral multiple of 360° and both angles are placed in standard position, how will the terminal sides of the two angles be situated with respect to each other?

4. If two angles, placed in standard position, have the same terminal side, what is the relation between them?

5. If the sum of two angles is an integral multiple of 360°, how will the terminal sides of the two angles be situated with respect to each other, both angles being placed in standard position?

62. The notion of the trigonometric functions of a general angle. Having formulated the notion of a *general angle*, it becomes necessary to revise our definitions of the trigonometric functions, since our original definitions are applicable to acute angles only. To be sure, we have already made some progress in this direction by defining the functions of an obtuse angle. (See Arts. 40 and 42.) But those definitions were provisional, and it will be advisable to reopen the whole question, so as to gain a larger and more adequate point of view.

Our new concept of an angle, as a measure for the amount of rotation of a line, practically forces upon us the following considerations which automatically suggest the new definitions of the trigonometric functions.

Let us draw a positive acute angle θ (see the Greek alphabet on page 132) in its standard position (Fig. 74 *a*). Let us choose a point P anywhere on its terminal side and from P drop a perpendicular PM to the initial side. Then, in accordance with the definitions of the functions of an acute angle, we have

$$(1) \qquad \sin\theta = \frac{MP}{OP}, \quad \cos\theta = \frac{OM}{OP}.$$

Let us now think of the angle θ as growing. Nothing remarkable happens until θ reaches 90°. At that moment, and as the motion continues, our original definitions cease to be applicable, because the right triangle POM, of which θ is an *interior* angle, ceases to exist. But we may think of PMQ (in Fig. 74 *a*) as a plumb line attached to a point P on the moving terminal side of the angle. If there is no obstacle at O, this line, remaining always vertical, will pass from the right to the left of O as θ grows from an acute into an obtuse angle (see Fig. 74 *b*), and the line-segment OM will change its direction. To indicate this change of direction of OM, we represent OM by a positive number in Fig. 74 *a* and by a negative number in Fig. 74 *b*. If we count the distance OP (which does not change) as positive, in all positions of the moving line, and if we retain equations (1) as definitions for $\sin\theta$ and $\cos\theta$, we see that $\cos\theta$ becomes *negative* when θ becomes obtuse. The line-segment PM does not change its direction until θ grows beyond 180°. Therefore the sine of an obtuse angle, like that of an acute angle, is positive. The sine of an angle, however, becomes negative when the angle lies between 180° and 360°.

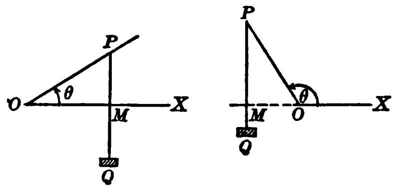

FIG. 74 *a* FIG. 74 *b*

63. Rectangular coördinates. All of these things may be stated more briefly by the introduction of *rectangular coördi-*

nates, a notion of utmost importance, not merely in trigonometry, but in other branches of mathematics.

Let us draw two lines, unbounded in length and perpendicular to each other. We shall usually think of one of them as horizontal and call it the **x-axis**, and call the other, which is vertical, the **y-axis**. The point O, in which the two axes intersect, is called the **origin of coördinates**.

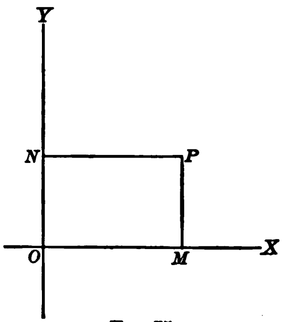

Fig. 75

We adopt a unit of length, and denote the distances from any point P to these two axes by x and y respectively. In Fig. 75 we have

$$NP = OM = x, \quad MP = ON = y,$$

where the notation is chosen in such a way that x is measured on or parallel to the x-axis, and y on or parallel to the y-axis.

We call x the **abscissa** and y the **ordinate** of the point P. Both numbers together are called the **coördinates** of P.

If we take into account only the magnitudes and not the directions of the lines OM, ON, etc., that is, if x and y are regarded as numbers without sign, there will be four points which have the same coördinates.

For instance, the points P, P', P'', P''', in Fig. 76, would all correspond to $x = 3$, $y = 2$.

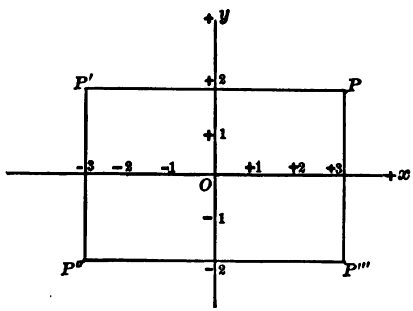

Fig. 76

In order to avoid this inconvenience, we introduce the convention that the abscissas of all points to the right of the y-axis shall be positive, and of those to the left negative; that the ordinates of all points above the x-axis shall be positive, and of those below negative.

The coördinates of the four points in Fig. 76 are now different from each other.

The coördinates of P are $x = +3$, $y = +2$,
The coördinates of P' are $x = -3$, $y = +2$,
The coördinates of P'' are $x = -3$, $y = -2$,
The coördinates of P''' are $x = +3$, $y = -2$.

The positive directions of the x- and y-axes, which have now been defined, *will hereafter be indicated by a plus sign* (as in Fig. 76).

The distance from the origin O to any point P is usually denoted by r and is called the **radius vector** of that point. We shall always regard the radius vector as *positive*, and clearly we shall always have (see Fig. 77),

$$r = +\sqrt{x^2 + y^2}.$$

Let us think of OP (Fig. 77) as rotating around O as a center. If OP originally coincides with the positive x-axis, it will require a counterclockwise rotation of 90°, or a clockwise rotation of 270°, to bring it into coincidence with the positive y-axis. We naturally think of the numerically smaller angle first, and define the *positive sense of rotation* to be *that one* which enables us to turn the *positive* x-axis into the position of the *positive* y-axis by means of a rotation of *only 90°*. But this implies that the *positive direction of rotation is counterclockwise*. (Cf. Art. 60.)

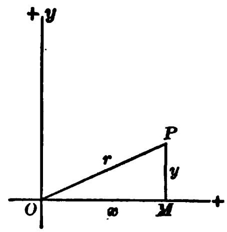

FIG. 77

EXERCISE XXXIII

Plot the points whose coördinates are given in Exs. 1 to 5. Find, by *measurement*, to the nearest degree for the angles and to the nearest tenth of a unit for the distances, the radius vector of each point and the positive angle which it makes with the positive x-axis. Find the same results by *calculation*, making use of three-place tables.

1. $x = +3, y = +4.$
2. $x = +12, y = -5.$
3. $x = -2.4, y = +5.5.$
4. $x = -2.6, y = -2.1.$
5. $x = +1.27, y = -2.18.$

In the following examples, r denotes the radius vector of a point P, and θ the positive angle which this radius vector makes with the positive direction of the x-axis. Plot the points. Find, by *measurement* to the nearest tenth of a unit, the abscissas and ordinates of these points. Find the same results by *calculation* with three-place tables.

6. $r = 2, \theta = 30°$.

7. $r = 5, \theta = 135°$.

8. $r = 3, \theta = 210°$.

9. $r = 4, \theta = 285°$.

10. $r = 2.56, \theta = 310° 20'$.

64. Definition of the trigonometric functions of a general angle. We are now in a position to give the definitions of the functions of a general angle in a compact manner.

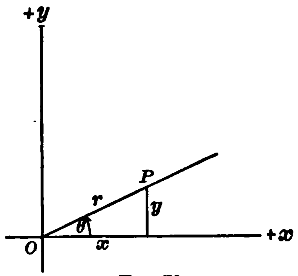

FIG. 78

Place the angle θ in its standard position; that is, with its vertex on the origin and its initial side on the positive x-axis of a system of rectangular coördinates. (See Fig. 78, where θ is a positive acute angle.) *Pick out a point P, different from the origin, anywhere on the terminal side of the angle. Then we adopt the following definitions:*

The **sine** of angle θ $= \dfrac{\text{ordinate of } P}{\text{radius vector of } P}$, $\quad$ **sin** $\theta = \dfrac{y}{r}$.

The **cosine** of angle θ $= \dfrac{\text{abscissa of } P}{\text{radius vector of } P}$, $\quad$ **cos** $\theta = \dfrac{x}{r}$.

The **tangent** of angle θ $= \dfrac{\text{ordinate of } P}{\text{abscissa of } P}$, $\quad$ **tan** $\theta = \dfrac{y}{x}$.

The **cotangent** of angle θ $= \dfrac{\text{abscissa of } P}{\text{ordinate of } P}$, $\quad$ **cot** $\theta = \dfrac{x}{y}$.

The **secant** of angle θ $= \dfrac{\text{radius vector of } P}{\text{abscissa of } P}$, $\quad$ **sec** $\theta = \dfrac{r}{x}$.

The **cosecant** of angle θ $= \dfrac{\text{radius vector of } P}{\text{ordinate of } P}$, $\quad$ **csc** $\theta = \dfrac{r}{y}$.

As Fig. 78 shows, these definitions reduce to the familiar definitions of Art. 7 if θ is an acute angle. In the case of an obtuse angle, they give results which agree with the definitions of Arts. 40, 42, and 43. But in our present definitions, θ is not restricted either in magnitude or sign; it may be a positive or negative angle of any magnitude. The quantities x and y may be positive, zero, or negative, but r is always positive.

EXERCISE XXXIV

Construct carefully the following angles and, by measurement, find approximate values of the six trigonometric functions, correct to two significant places, paying particular attention to their signs.

1. $25°$. **2.** $320°$. **3.** $110°$. **4.** $-130°$. **5.** $+725°$. **6.** $-10°$.

7. In Art. 11, the exact values of the functions of 30°, 45°, and 60° were expressed by means of radicals. In a similar way find the values of the functions of the following angles:

120°, 135°, 150°, 210°, 225,° 240°, 300°, 315°, 330°.

8. What are the signs of the trigonometric functions of the following angles:

$150°,\ 320°,\ 1000°,\ -625°.$

Find, by construction and measurement to the nearest degree, the values of the angles for which

9. $\sin\theta = -\frac{4}{5},\ \cos\theta = +\frac{3}{5}.$

10. $\tan\theta = 1,\quad \cos\theta = -\dfrac{1}{\sqrt{2}}.$

11. Show that the values of the trigonometric functions of a general angle, as given by the definitions of Art. 64, will not be changed if, instead of the point P, any other point P' on the terminal side of the angle be chosen.

65. Discussion of the exceptional cases. Each of the trigonometric functions is defined in Art. 64 *formally*, as a quotient of two numbers. This formal definition will have a real significance whenever the two numbers actually *have* a quotient. Now we know from Algebra that two numbers, D (the dividend) and d (the divisor), always have a unique

quotient q if the divisor is ***different from zero.*** That is, there exists a number q such that

$$\frac{D}{d} = q, \tag{1}$$

or what amounts to the same thing, such that

$$D = dq, \tag{2}$$

whenever d is different from zero.

Now, let us discuss the case where d (the divisor) is equal to zero, while D (the dividend) is not. In this case there exists *no* number q which satisfies equation (2). For this equation now becomes

$$D = 0 \cdot q, \tag{3}$$

and its right member is equal to zero no matter what number we substitute for q, while its left member is, by hypothesis, different from zero. Consequently it involves a contradiction to assume that a number has been obtained by dividing another by zero, and **the operation of dividing by zero is therefore excluded from Algebra.**

The formal definitions of Art. 64 involve divisions by x, y, and r, and therefore lose their significance in any case in which one of these divisors is equal to zero. Now $r = OP$ is the radius vector of a point P which may be chosen anywhere on the terminal side of the angle except at O. (Compare the wording of the definitions in Art. 64.) Therefore r is never equal to zero. From this fact and the equation

$$x^2 + y^2 = r^2, \tag{4}$$

we conclude further that x and y cannot both be equal to zero at the same time.

It will now be clear that the formal definitions of Art. 64 fail to provide the symbols

$$\begin{matrix} \tan 90°, & \sec 90°, & \tan 270°, & \sec 270°, \\ \cot 0°, & \csc 0°, & \cot 180°, & \csc 180°, \end{matrix} \tag{5}$$

with any actual meaning. But they *do* define each of the six trigonometric functions of all positive angles less than 360° with the eight exceptions just mentioned.

If the angle θ is greater than 360°, or if it is negative, other exceptional cases appear. But their relation to the eight exceptional cases (5) is so simple that we may leave it to the student to complete this discussion.

Although the tangent of 90° is not defined, our definitions are clearly applicable to the tangent of an angle θ which differs from 90° by the slightest conceivable amount. Let us see how the tangent of an angle θ behaves when θ approaches 90° as a limit. We have, in Fig. 79,

$$\tan\theta = \frac{y}{x} = \frac{MP}{OM},$$

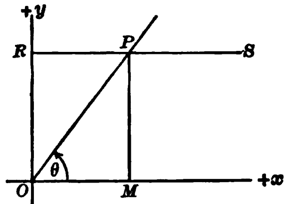

FIG. 79

where P may be any point different from O on the terminal side of the angle. For our present purpose it will be convenient to select the point P in the following manner. Draw a line RS parallel to the x-axis at any convenient distance from OM, and let P be the intersection of the terminal side of the angle θ with this fixed line. Then, as θ approaches 90°, $MP = y$ will remain constant, while $OM = x$ approaches the limit zero. The quotient $\frac{y}{x} = \tan\theta$ will therefore become larger and larger. Since $OM = x$ may be made as *small* as we please by taking the angle $\theta = MOP$ close enough to 90°, while the ordinate MP always remains the same, we see that the quotient can be made as *large* as we please. In other words, the angle θ can be made to differ so little from 90° that its tangent will become larger than any number whatsoever. This is what is meant by the statement that $\tan\theta$ *becomes infinite* when θ approaches 90° as a limit. We sometimes express this same statement by writing

$$\tan 90° = \infty, \tag{6}$$

a symbolic equation which should be interpreted as a shorthand account of the situation which has just been described. It is *not* a definition of tan 90°. For ∞ is not a number, and

the symbolic equation (6) is not at all concerned with what happens to tan θ when θ is equal to 90°. It merely tells us, in symbolic form, what happens when θ *approaches* 90° as a limit; namely, that tan θ then increases *without bound.*

In the preceding discussion we considered a variable angle MOP which approached 90° as a limit. In order to see what happened to its tangent, we chose the point P on the terminal side of the angle in such a way that its distance y from the x-axis remained constant.

We may obtain the same result in a slightly different way, which may, to some students, appear more conclusive. Let the angle MOP approach 90° as before. Then (Fig. 80),

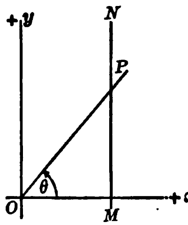

FIG. 80

$$\tan MOP = \frac{MP}{OM} = \frac{y}{x}.$$

Let us, this time, choose the point P on the terminal side of the angle in such a way that its distance x from the y–axis remains unchanged while θ approaches 90° as a limit. It is evident from the figure that $MP = y$ will then increase without bound. Therefore, we obtain again the result that tan $\theta = y/x$ *becomes infinite* when θ approaches 90° as a limit.

So far we have tacitly assumed that θ is an *acute* angle *increasing* toward 90° as a limit. What happens when θ starts as an *obtuse* angle to *decrease* toward 90° as a limit?

Since the tangent of any angle between 90° and 180° is negative, an argument precisely similar to that just carried out shows that the numerical value of tan θ again grows beyond all bounds when θ approaches 90°, remaining, however, always negative. We see therefore that the following statements are both true:

1. When θ is *acute* and *increases* toward 90° as a limit, tan θ, remaining always *positive*, grows numerically beyond bound.

2. When θ is *obtuse* and *decreases* toward 90° as a limit, tan θ, remaining always *negative*, grows numerically beyond bound.

These two statements are frequently summed up in the symbolic formula

(7) $$\tan 90° = \pm\infty.$$

A precisely similar discussion will show that $\tan\theta$ again becomes infinite when θ approaches 270°, that $\sec\theta$ becomes infinite when θ approaches 90° or 270°, and that $\cot\theta$ and $\csc\theta$ become infinite when θ approaches either 0° or 180°. The functions $\sin\theta$ and $\cos\theta$ are always finite.

66. The four quadrants. The x- and y-axes divide the plane into four portions called *quadrants*. The quadrant bounded by the positive x- and y-axes is usually called the *first quadrant*. If we start from the first quadrant and describe a path around the origin in the counterclockwise direction, we traverse in order the 1st, 2d, 3d, and 4th quadrants.

An angle is said to be in the first, second, third, or fourth quadrant according to the quadrant in which its terminal side falls when the angle is in its standard position, that is, with its initial side upon the positive x-axis.

The *cardinal angles* 0°, 90°, 180°, 270°, etc., may be regarded as belonging to either one of the two quadrants upon whose boundaries they lie.

The following table gives the signs of the trigonometric functions of an angle in the various quadrants:

	I	II	III	IV
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent . . .	+	−	+	−
Cotangent . . .	+	−	+	−
Secant	+	−	−	+
Cosecant . . .	+	+	−	−

EXERCISE XXXV

1. Prove each of the following symbolic statements and explain its significance in words.

$$\cot 0^\circ = \pm \infty, \quad \tan 90^\circ = \pm \infty, \quad \cot 180^\circ = \pm \infty, \quad \tan 270^\circ = \pm \infty,$$
$$\csc 0^\circ = \pm \infty, \quad \sec 90^\circ = \pm \infty, \quad \csc 180^\circ = \pm \infty, \quad \sec 270^\circ = \pm \infty.$$

2. Show that the numerical value of the sine or cosine of an angle can never exceed unity.

3. Show that D/d may have any value whatever if D and d are both equal to zero, and hence that the symbol $\frac{0}{0}$ is wholly indeterminate. Why can no one of the trigonometric ratios ever have this form?

4. Determine the quadrants of the following angles and the signs of their trigonometric functions.

$$325^\circ, 710^\circ, 1045^\circ, 609^\circ, 412^\circ, -52^\circ.$$

5. In what quadrant is an angle if its sine and cosine are both positive? If its sine is positive and its tangent negative? If its secant and tangent are both positive?

6. If we know that the sine and cosine of an angle have the same sign, what can we say about the quadrant of the angle?

7. Is there an angle whose tangent is positive and whose cotangent is negative?

8. If we are told that the tangent and cotangent of an angle are both positive, does this enable us to determine the quadrant of the angle?

67. General character of the trigonometric functions. Their periodicity. We are now in a position to understand how the functions change with the angle. For the purposes of this discussion it will be convenient to think of the radius vector r as constant. This means that the point P, which according to our definition must be selected on the terminal side of the angle, describes the circumference of a circle as θ changes from 0° to 360°.

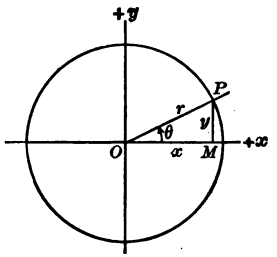

Fig. 81

(See Fig. 81.) It is easy to verify the following statements by reference to the figure:

As θ increases from 0° to 90°, $\sin\theta$ increases from 0 to 1.
As θ increases from 90° to 180°, $\sin\theta$ decreases from 1 to 0.
As θ increases from 180° to 270°, $\sin\theta$ decreases from 0 to -1.
As θ increases from 270° to 360°, $\sin\theta$ increases from -1 to 0.

It is also evident that the function $\sin\theta$ repeats its values in exactly the same order if P moves around the circumference a second, third, $\cdots$ nth time. The same thing is true of the other trigonometric functions, a very important fact which may be expressed as follows:

Each of the six trigonometric functions is ***periodic*** *and its period is equal to* 360°. *That is, each of these functions repeats its values at intervals of* 360°, *so that*

$$\sin(\theta + n\cdot 360^\circ) = \sin\theta,\quad \cos(\theta + n\cdot 360^\circ) = \cos\theta, \text{ etc.},$$

where n is any positive or negative integer or zero.

The behavior of each of the six functions in the neighborhood of each of the four cardinal angles may be recapitulated for convenience of reference in the following table:

	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
0°	0	$+1$	0	$\mp\infty$	$+1$	$\mp\infty$
90°	$+1$	0	$\pm\infty$	0	$\pm\infty$	$+1$
180°	0	-1	0	$\mp\infty$	-1	$\pm\infty$
270°	-1	0	$\pm\infty$	0	$\mp\infty$	-1

The student should use this table to describe in words the variation of each of the six functions as θ changes from 0° to 360°.

EXERCISE XXXVI

Discuss the variation of the following functions as θ varies from 0° to 360°.

1. $\cos\theta$.
2. $\tan\theta$.
3. $\cot\theta$.
4. $\sec\theta$.
5. $\csc\theta$.
6. $\sin 2\theta$.
7. $2\sin\theta$.
8. $\sin 3\theta$.
9. $\sin 4\theta$.
10. $\tan 4\theta$.
11. $\sin(-\theta)$.
12. $\cos(-\theta)$.
13. $\sin\frac{1}{2}\theta$.
14. $\sin\frac{1}{3}\theta$.
15. $\sin(\theta + 25^\circ)$.

68. Relations between the trigonometric functions of a general angle. The relations which we found in Art. 9, between the functions of an acute angle, still hold without alteration for an angle of any magnitude. In fact the equation between the abscissa, ordinate, and radius vector of a point P, that is,
$$x^2 + y^2 = r^2,$$
is true, no matter in what quadrant the point P may be situated.

This is due to the fact that only the squares of x, y, and r occur in this relation.

If we divide both members of the above equation by r^2, we find
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1.$$
But, by the definitions of Art. 64, we have, in all quadrants,
$$\frac{x}{r} = \cos\theta, \quad \frac{y}{r} = \sin\theta,$$
so that the preceding equation becomes

(1) $$\boldsymbol{\sin^2\theta + \cos^2\theta = 1.}$$

Since we have, by definition,
$$\sin\theta = \frac{y}{r}, \qquad \csc\theta = \frac{r}{y},$$
$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x},$$
$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{x}{y},$$
we find at once

(2) $$\boldsymbol{\sin\theta\csc\theta = 1, \quad \cos\theta\sec\theta = 1, \quad \tan\theta\cot\theta = 1.}$$

We have also
$$\tan\theta = \frac{y}{x} = \frac{y/r}{x/r} = \frac{\sin\theta}{\cos\theta},$$
which, combined with (2), gives the further relations

(3) $$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}.}$$

If we divide both members of (1) by $\cos^2\theta$ and make use of (2) and (3), we find

(4) $$\boldsymbol{1 + \tan^2\theta = \sec^2\theta.}$$

In a similar fashion, if we divide both members of (1) by $\sin^2\theta$, we see that

$$(5) \qquad 1 + \cot^2\theta = \csc^2\theta.$$

When the angle θ is acute, all of its functions are positive. Consequently if one of its functions is given, all of the others may be found without ambiguity by means of the above relations. (Cf. Art. 9 and Exercise VI, Exs. 7–12.)

But if we do not know in what quadrant an angle lies and are given the value of merely one of its functions, the angle itself and its other functions are not determined uniquely.

If we are told, for instance, that $\sin\theta = \frac{1}{2}$, equation (1) only tells us that

$$\cos^2\theta = 1 - (\tfrac{1}{2})^2 = \tfrac{3}{4};$$

so that

$$\cos\theta = \pm \tfrac{1}{2}\sqrt{3},$$

where either sign may be taken. In fact there *are* two angles between 0° and 360° whose sines are equal to $\frac{1}{2}$; namely, 30° and 150°. We may distinguish between them by stating whether the cosine is positive or negative.

EXERCISE XXXVII

Find the other functions of the angle θ as determined by each of the following conditions:

1. $\sin\theta = -\frac{1}{2}$ and θ is in the third quadrant.

2. $\sin\theta = -\frac{1}{2}$ and θ is in the fourth quadrant.

3. $\tan\theta = +2$ and θ is in the third quadrant.

4. $\cot\theta = -3$ and $\sin\theta$ is positive.

5. $\sec\theta = +2$ and $\tan\theta$ is negative.

6. Find the values of the other functions if $\sin\theta = a$. Are all possible values of a admissible? State a reason for your answer.

7. If $\tan\theta = m$, find the values of the other functions.

8. If $\sec\theta = k$, find the values of the other functions. Are all values of k admissible in this problem? State a reason for your answer.

69. Trigonometric identities which involve functions of a single angle. By means of the relations of the preceding article, an expression which involves the trigonometric functions of an angle θ may be written in a great many different forms. It is often important to be able to recognize that two trigonometric expressions, although different in form,

are really identical. This may frequently be done by inspection. In more complicated cases it is advisable to express each of the two quantities, whose identity we wish to establish, in terms of some one of the six functions (the sine, for example). It will then become evident as a mere matter of algebra whether or not the two quantities are really identical.

EXERCISE XXXVIII

1. Show that $\sec\theta - \tan\theta \sin\theta = \cos\theta$, for all values of θ for which $\tan\theta$ and $\sec\theta$ are defined. (See p. 142.)

Solution. We have, for all values of θ, for which $\tan\theta$ and $\sec\theta$ are defined,

$$\sec\theta - \tan\theta\sin\theta = \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta}\sin\theta = \frac{1-\sin^2\theta}{\cos\theta} = \frac{\cos^2\theta}{\cos\theta} = \cos\theta,$$

which proves the truth of the original assertion.

2. Prove that $\tan A + \cot A = \sec A \csc A$ is an identity.*

Solution. Denote the quantity on the left member by L and that on the right member by R. Then

$$L = \tan A + \cot A = \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} = \frac{\sin^2 A + \cos^2 A}{\sin A \cos A} = \frac{1}{\sin A \cos A},$$

$$R = \sec A \csc A = \frac{1}{\cos A}\frac{1}{\sin A} = \frac{1}{\sin A \cos A}.$$

Therefore $L = R$; that is,

$$\tan A + \cot A = \sec A \csc A. \qquad \text{Q. E. D.}$$

Prove that the following statements are identities:

3. $\cos\theta\tan\theta = \sin\theta$.

4. $\sin\phi\cot\phi = \cos\phi$.

5. $\sin^2\theta + \sin^2\theta\tan^2\theta = \tan^2\theta$.

6. $\cos^2 A - \sin^2 A = 1 - 2\sin^2 A$.

7. $\cos^2 A - \sin^2 A = 2\cos^2 A - 1$.

8. $(\csc^2\theta - 1)\sin^2\theta = \cos^2\theta$.

9. $1 + \tan^2\theta = \dfrac{1}{1-\sin^2\theta}$.

10. $\cos^4\theta - \sin^4\theta = 2\cos^2\theta - 1$.

11. $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$.

12. $\dfrac{\sec\theta\cot\theta - \csc\theta\tan\theta}{\cos\theta - \sin\theta} = \csc\theta\sec\theta$.

13. $\dfrac{\cos\theta\cot\theta - \sin\theta\tan\theta}{\csc\theta - \sec\theta} = 1 + \sin\theta\cos\theta$.

14. $\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}} = \sec\theta - \tan\theta$, if θ is an acute angle.

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*In other words, show that the left member is equal to the right member for all values of A for which the functions $\tan A$, $\cot A$, $\sec A$, $\csc A$ have been defined; that is, for all values of A except $A = 0°, 90°, 180°, 360°$, etc.

CHAPTER X

GRAPHIC REPRESENTATIONS OF THE TRIGONOMETRIC FUNCTIONS

70. Line representation of the trigonometric functions. The trigonometric functions were *defined* as *ratios* or *abstract* numbers, not as lines. (See Art. 7.) This does not, however, preclude the possibility of *representing* them as lines. *An abstract or concrete quantity of any kind may be represented as a line-segment, by choosing arbitrarily a certain line-segment to represent a unit of the same kind.*

Thus we may represent as lines the populations of the various states of the Union, taking a line-segment one inch long to represent a population of 1,000,000. The populations of New York and Illinois will then be represented by line-segments 9.11 and 5.64 inches in length respectively. Thus, although a population obviously is not a line-segment, it may be *represented* by a line-segment. In the same way we may represent the values of the trigonometric functions by lines, although they are not lines, but abstract numbers.

The following is a convenient method for obtaining a representation of the values of the trigonometric functions as line-segments.

We construct a circle with the origin of coördinates as center. An angle whose vertex is at the center of this circle will subtend an arc whose numerical measure, in degrees, minutes, and seconds, is equal to that of the angle. We may therefore speak indifferently either about the functions of the *angle* or of the functions of the *arc*. The point in which the initial side of the angle meets the circle is called the *origin of the arc*. The point in which the terminal side of the angle meets the circle is called the *terminus, or the end of the arc.*

If an angle is placed in its standard position, the origin of the subtended arc will be at A, the point in which the positive x-axis meets the circle. We shall call this point the *primary origin of arcs*. The point B, in which the positive y-axis intersects the circle is called the *secondary origin of arcs*.

Let us choose any convenient unit of length, say an inch, and let us agree to measure all distances in terms of this unit. We then construct the circle, the so-called *unit circle*, whose center is at the origin of coördinates and whose radius is equal to the unit of length.

In Fig. 82, let $AOQ = \theta$ be any angle in its standard position, and AP the arc which it subtends on the unit circle. Then

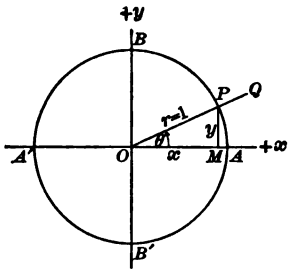

FIG. 82

(1) $$\sin\theta = \frac{y}{r} = \frac{y}{1} = y = MP,$$

$$\cos\theta = \frac{x}{r} = \frac{x}{1} = x = OM,$$

since the distance $r = OP$ is equal to the unit of length, so that $r = 1$. Now $x = OM$ and $y = MP$ are the coördinates of P, the terminus of the arc AP. Consequently we may express our result as follows:

If any angle θ is placed in its standard position, the value of its sine is equal, in magnitude and sign, to the ordinate of the terminus of the arc which the angle subtends on the unit circle. The value of its cosine is equal to the abscissa of the terminus of this arc.

In order to find a line representation for $\tan\theta$ and $\cot\theta$, we draw tangents to the unit circle at A and B and denote by T and T' the points in which the terminal side of the angle intersects these two tangents. (See Fig. 83.)

Then we find

$$\tan\theta = \frac{AT}{OA} = \frac{AT}{1} = AT,$$

$$\cot\theta = \tan BOT' = \frac{BT'}{OB} = \frac{BT'}{1} = BT',$$

since the radius of the circle

$$OA = OB = 1.$$

If θ is an obtuse angle, OP will have to be prolonged backward in order to intersect the tangent at A. Moreover the point of intersection T will then be below A. Now the tangents BT' and AT are parallel to the x- and y-axes respectively. Let us agree to give signs to the line-segments measured on these two tangents as though they were abscissas or ordinates of a point. That is, let AT be positive or negative according as T is above or below A, and let BT' be positive or negative according as T' is to the right or left of B. This convention is indicated in Fig. 83 by the two + signs at the ends of the two tangents. The student may now verify that the equations

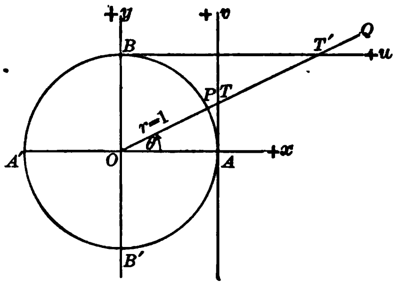

FIG. 83

$$(2) \qquad \tan\theta = AT, \quad \cot\theta = BT',$$

which we have obtained from Fig. 83 in the case of an acute angle, will give correct results in *magnitude* and *sign*, no matter in what quadrant the angle θ may happen to fall.

We may formulate our results as follows:

If any angle θ is placed in its standard position, the value of its tangent is equal, in magnitude and sign, to the ordinate of the point in which the terminal side of the angle, prolonged backward if necessary, intersects the tangent to the unit circle at the primary origin of arcs.

The cotangent of the angle is equal, in magnitude and sign, to the abscissa of the point in which the terminal side of the angle, prolonged backward if necessary, intersects the tangent to the unit circle at the secondary origin of arcs.

Referring once more to Fig. 83, we have

$$(3)\qquad \begin{aligned} \sec\theta &= \frac{OT}{OA} = \frac{OT}{1} = OT, \\ \csc\theta = \sec BOT' &= \frac{OT'}{OB} = \frac{OT'}{1} = OT'. \end{aligned}$$

These line representations for the secant and cosecant will hold, in magnitude and sign, not merely for acute angles, but for angles in any quadrant, if we agree to make the following conventions in regard to sign. OT shall be positive if T is on the same side of O as P, *i.e.* if T is on the terminal side of the angle θ. OT shall be negative if T is on the terminal side of the angle θ prolonged backward. OT' shall be positive or negative according as T' falls on the terminal side of the angle θ or on the terminal side prolonged backward.

We leave it to the student as an exercise to verify these statements in detail and to formulate the contents of equations (3) in words.

Figures 84 to 87 illustrate the line representation of the

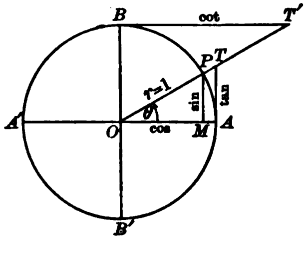

Fig. 84

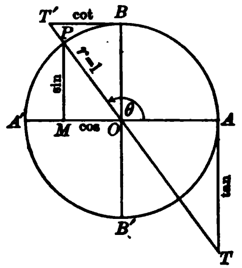

Fig 85

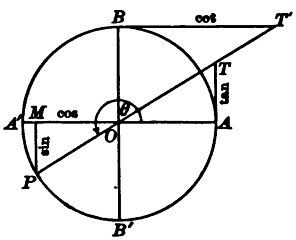

Fig. 86

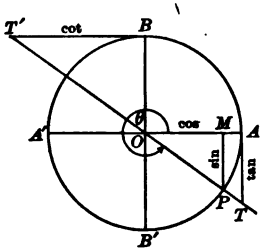

Fig. 87

trigonometric functions for an angle in each one of the four quadrants. In each of these figures

$$MP = \sin\theta,\quad AT = \tan\theta,\quad OT = \sec\theta,$$
$$OM = \cos\theta,\quad BT' = \cot\theta,\quad OT' = \csc\theta.$$

These line representations of tan θ and sec θ suffice to explain why the names tangent and secant were chosen for these functions. The word sine is not capable of such a simple explanation and has a long and complicated history.

The Greeks did not use the six functions which we have introduced. In place of the sine of an angle they made use of the chord PQ, subtended by the angle POQ, on a circle of known radius. (See Fig. 88.) If the circle has a unit radius, Fig. 88 shows that this chord PQ is equal to twice QR, or

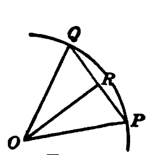

FIG. 88

$$PQ = 2\sin\tfrac{1}{2}POQ.$$

Thus the chord, used by the Greeks, is essentially twice the sine of half the angle.

ĀRYABHATA, a famous Hindoo mathematician (born 476 A.D.), was apparently the first to introduce the sine of the angle in place of the chord, and he, quite naturally, spoke of it as the half-chord or *jyā-ardhā*, where *jyā* is the Sanskrit for chord or bowstring and *ardhā* for one half. For the sake of brevity the adjective *ardhā* was soon omitted and the sine was called simply *jyā*.

The *Arabs*, who far more than any other people cultivated the sciences during the Middle Ages, took over this word from the Hindoos, but changed its spelling to *jība*, so as to make the spelling accord with the pronunciation in the sense of their own language.

But in written Arabic the consonants only are represented by definite characters, the vowels being merely indicated by dots which are frequently omitted altogether. As a consequence of this practice, the Hindoo word *jība* was soon corrupted into *jaīb*, a genuine Arabic word meaning *bosom, heart,* or *pocket* according to the context.

In the twelfth century, when the Arabic texts were translated into Latin, the word *jaīb* was translated literally by the Latin word *sinus* meaning bosom.

Thus, a foreign word was first converted by the Arabs into a word of their own language having a similar sound but an entirely different meaning, and later this Arabic word was translated literally into Latin. Of course the derivation of the English word *sine* from *sinus* is obvious.

71. Graphs of functions, a number of whose numerical values are given. There is a second way of representing

graphically the values of the trigonometric functions, which is even more important than that which has just been discussed. For it gives us, in a still more vivid fashion, a picture of all the most essential properties of these functions; and it has the further advantage of being applicable, not merely to the trigonometric functions, but to all of the other functions which naturally arise in pure and applied mathematics. In order to lead the student to appreciate fully the power of this new method, we shall first illustrate it by a number of examples taken from fields other than trigonometry.

x = Date	y = Population
1790	3,929,214
1800	5,308,483
1810	7,239,881
1820	9,638,453
1830	12,860,692
1840	17,063,353
1850	23,191,876
1860	31,443,321
1870	38,558,371
1880	50,155,783
1890	62,947,714
1900	75,994,575
1910	91,972,266

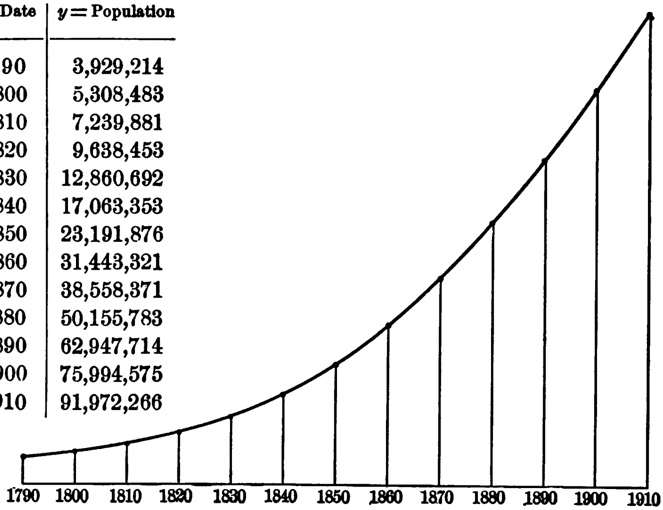

FIG. 89

The population of the United States is determined every ten years by a national census. The table in the margin gives the results of these censuses. It is customary to represent the contents of this table graphically by laying off the dates horizontally (*i.e.* as abscissas), and erecting for each of these dates a vertical line (ordinate), which shall give by its length in terms of an appropriately chosen unit the population at that time. Figure 89 gives such a graphic representation of the facts contained in this population table and presents these facts in a more easily intelligible form than the table itself. Moreover if we join the endpoints of the ordinates by a smooth curve (the population curve),

we may draw some fairly reliable conclusions as to the state of the population in the years 1805, 1815, etc., in which no census was taken.

If we plot the population curves of two or more countries upon the same sheet, a great many interesting matters may be brought out by comparison.

Clearly we may adopt such a graphic method, whenever we have a table giving a relation between two variables; that is, a table which shows that to certain numerical values of a first quantity x there correspond certain numerical values of a second quantity y.

EXERCISE XXXIX

1. The following table gives the population of the cities of New York, Chicago, and Philadelphia for the years named:

	1850	1860	1870	1880	1890	1900	1910
New York .	515,547	805,651	942,292	1,206,299	1,515,301	3,437,202	4,766,883
Chicago . .	28,269	109,206	298,977	503,298	1,099,850	1,698,572	2,185,283
Philadelphia	340,045	585,529	674,022	847,170	1,046,964	1,293,697	1,549,008

Make a graph illustrating this information,* and from the graph find the probable population of each of these cities in 1908.

2. Make a population table and a population curve for the city and state in which you live.

3. Let the student provide himself with a railroad time-table, giving the names of the various stations, their distances from the starting point, and the times at which a certain train leaves these stations. Draw a distance-time diagram for one or several trains, plotting the times as abscissas and the distances as ordinates.

4. On April 3, 1912, the following temperatures were observed in Chicago:

3 A.M.	32°	11 A.M.	38°	7 P.M.	34°
4 A.M.	32°	12 M.	39°	8 P.M.	33°
5 A.M.	32°	1 P.M.	39°	9 P.M.	34°
6 A.M.	31°	2 P.M.	38°	10 P.M.	34°
7 A.M.	32°	3 P.M.	37°	11 P.M.	33°
8 A.M.	33°	4 P.M.	36°	12 P.M.	33°
9 A.M.	34°	5 P.M.	35°	1 A.M.	32°
10 A.M.	36°	6 P.M.	35°	2 A.M.	31°

Represent graphically.

* In all such work, involving plotting of curves, it is advisable to use cross-section paper.

72. Graphs of simple algebraic functions. The relation between the variables x and y, instead of being given by a table as in the previous examples, may be given by an equation. We may then use the equation for the purpose of constructing a table, and then draw a graph as before.

EXAMPLE 1. Find the graph of $y = 2x - 5$.

Solution. If we substitute $x = 0$ in the given equation, we find $y = -5$; for $x = 1$, we find $y = -3$; etc. We construct in this way the table printed in the margin. If we plot the points $x = -2$, $y = -9$; $x = -1$, $y = -7$; etc., obtained in this way, we find the points marked in Fig. 90 with a little cross. All of these points are found to be on a straight line.

x	y
-2	-9
-1	-7
0	-5
$+1$	-3
$+2$	-1
$+3$	$+1$
$+4$	$+3$
$+5$	$+5$

This observation makes it seem likely that *all* of the points whose coördinates satisfy the equation $y = 2x - 5$, not merely those which we happened to compute, are on this same straight line. It is not difficult to prove that this is so, but the proof will not be given here.

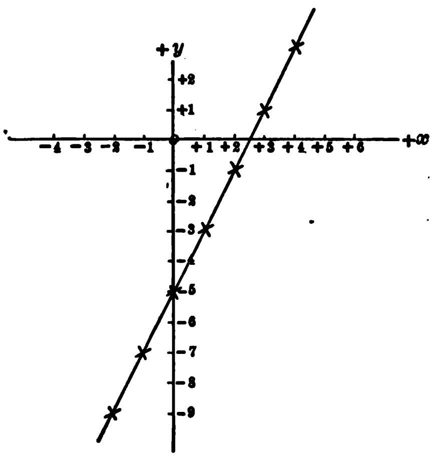

FIG. 90

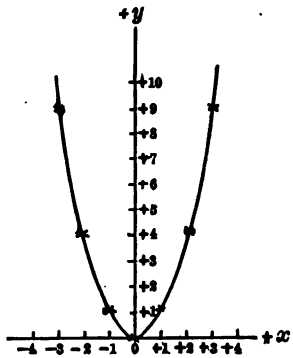

FIG. 91

EXAMPLE 2. Find the graph of $y = x^2$.

Solution. As before, we construct the table in the margin by computing the values of y which correspond to the values $x = -3, -2, -1, 0,$

x	y
-3	9
-2	4
-1	1
0	0
$+1$	1
$+2$	4
$+3$	9

$+1$, $+2$, $+3$. We then plot the points obtained in this way, the points marked with a little cross in Fig. 91, and unite them by a smooth curve.

The principle involved in these examples, that to every equation between two variables x and y there corresponds a curve and *vice versa*, is at the foundation of *Analytic Geometry*, which is one of the most important developments of modern mathematics. The great merit of having introduced this idea into mathematics is due to DESCARTES (1596–1650) and FERMAT (1601–1665).

EXERCISE XL

Find the graphs of the following equations:

1. $y = x$.
2. $y = -x$.
3. $y = -2x + 1$.
4. $y = 2x^2$.
5. $y = -x^2$.
6. $y = x^2 - 5$.
7. $y = x^2 - 5x$.
8. $y = 2x^2 - 3x + 1$.
9. $y = x^3$.
10. $y = \frac{1}{3}x^3 - 1$.
11. $y = \frac{1}{x}$.
12. $y = \frac{1}{x} - 1$.

73. Graphs of the trigonometric functions. We now proceed to apply this method to the relation

$$y = \sin x.$$

We choose an arbitrary line-segment on the x-axis to represent one degree and another arbitrary line-segment on the y-axis to represent the unit value of the sine, that is, the abstract number 1. If we are using millimeter paper, or a metric scale, it will be convenient to make a distance of one millimeter on the x-axis stand for one degree, and to measure the ordinates in terms of a unit 10 centimeters or 100 millimeters long. As this is a rather large scale it will probably be necessary to paste several sheets together in order to be able to construct the whole curve.

x	$y = \sin x$
0	0
10	17
20	34
30	50
40	64
50	76
60	87
70	94
80	98
90	100

From the table of natural sines we obtain the table in the margin, in which the values of the angle x as well as the corresponding values of $\sin x$ are expressed in millimetres in accordance with the adopted scale, which makes 1 mm. on the x-axis stand for 1°, and 100 mm. on the y-axis stand for the unit value of the sine. This table enables us to plot ten points of our curve representing ten values of the function $\sin x$ in the first quadrant. (See Fig. 92, which is a reduced copy of such a curve.)

The definition of the sine of a general angle (Art. 64) and the line representation of the sine in the unit circle (Art. 70), both show very clearly that two angles like 80° and 100°, or 70° and 110°, which

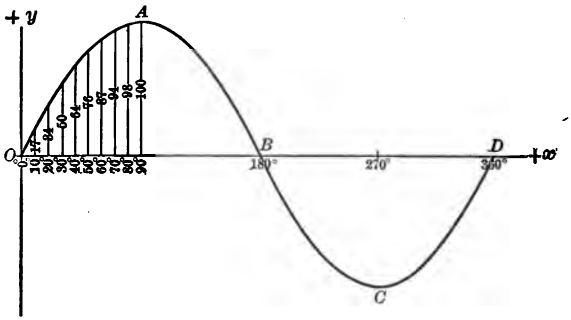

FIG. 92. — The Sine Curve

differ by the same amount from 90° but in opposite directions have the same sine. Consequently that portion AB of our curve, which represents the values of $\sin x$ for angles in the second quadrant, will be a symmetric counterpart of the first portion OA, which corresponds to angles in the first quadrant. (See Fig. 92.)

The unit circle also makes it evident that the sines of two angles which differ by 180° are numerically equal but opposite in sign. Consequently that portion BCD of our curve, which corresponds to angles in the third and fourth quadrants and all of whose ordinates are negative, may be obtained easily from the known part OAB. The parts OAB

and BCD of the curve are in fact congruent, but are situated on opposite sides of the x-axis.

We have already noted that the sine function repeats its values at intervals of 360°. It is a periodic function with a period of 360° (Art. 67). This manifests itself in the graph

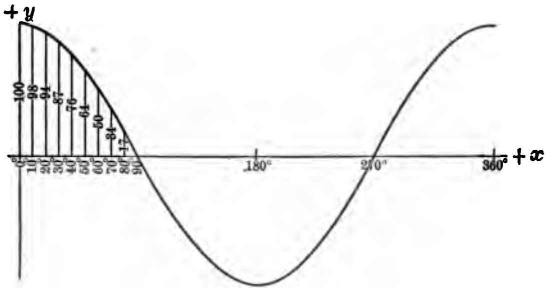

FIG. 93. — The Cosine Curve

by the fact that the picture in each of the intervals from 360° to 720°, etc., from − 360° to 0°, etc., is an exact copy of that piece of the curve which lies between 0° and 360°.

The curve obtained in this way from the sine function is called the **sine curve**.

The **cosine curve**, which is the graph of the function

$$y = \cos x,$$

is of the same general character as the sine curve. Its form is given in Fig. 93, and may be obtained by applying to the cosine function an argument exactly similar to that which has just been carried out for the sine.

We may apply the same method to the function

$$y = \tan x.$$

However, the graph obtained in this way differs very essentially from the sine and cosine curves.

In fact we know that when x approaches 90° from below, that is, if x assumes a succession of values like

$$89°, \ 89°.9, \ 89°.99, \ 89°.999, \text{ etc.},$$

the tangent of x, remaining always positive, will grow numerically beyond all bound. If on the other hand x approaches 90° from above, through a sequence of values like

$$91°, \quad 90°.1, \quad 90°.01, \quad 90°.001, \text{ etc.},$$

the tangent of x, remaining always negative, again grows numerically beyond all bound. (Cf. Art. 65.) We see, therefore, that the difference between the values of

$$\tan(90° + h) \text{ and } \tan(90° - h)$$

grows larger and larger as the angles

$$90° + h \text{ and } 90° - h$$

themselves come closer and closer together.

We express this by saying that *the function* $\tan x$ *is discontinuous for* $x = 90°$. The corresponding property of the graph is an interruption or break in the otherwise continuous curve. There are no such breaks in the sine or cosine curves. We can think of a material point (say the point of a lead pencil) as actually describing a sine curve without interruption. If we were to attempt to do the same thing for the tangent curve, we should have to interrupt the path of the point at $x = 90°$, at $x = 270°$, etc.

We meet here *the important distinction between continuous and discontinuous functions*, the precise formulation of which must be left to a later point in the student's career.

The tangent is, of course, a periodic function and repeats its values at intervals of 360°. But we may now observe that, unlike the sine or cosine, it repeats its values after the shorter interval of 180°. To recapitulate: *the tangent is a periodic function of period* 180°, *and is discontinuous for* $x = 90°$ *and for all values of* x *which differ from* 90° *by integral multiples of* 180°.

Figure 94 shows the form of the tangent curve.

EXERCISE XLI

1. Plot the curves $y = 2 \sin x$, $y = 3 \sin x$, $y = 4 \sin x$.

2. Plot the curves $y = \sin 2x$, $y = \sin 3x$, $y = \sin 4x$.

3. How are the curves of Exs. 1 and 2 related to the curve $y = \sin x$?

4. Show that the curve $y = \cot x$ is discontinuous for $x = 0°$, $180°$, $360°$, etc., and has the form indicated in Fig. 95.

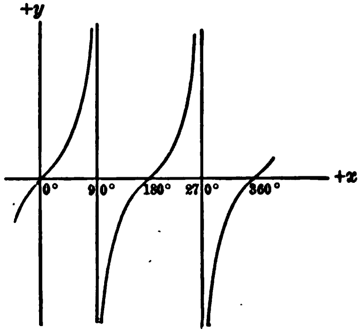

FIG. 94. — The Tangent Curve

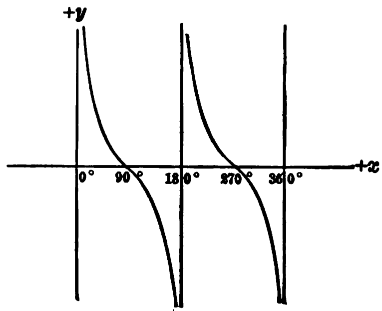

FIG. 95. — The Cotangent Curve

5. Show that the curves $y = \sec x$ and $y = \csc x$ have the forms indicated in Figs. 96 and 97.

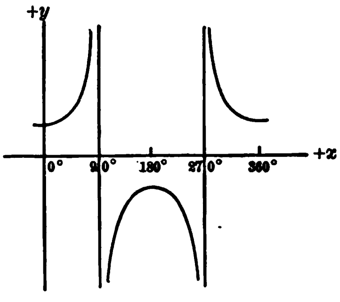

FIG. 96. — The Secant Curve

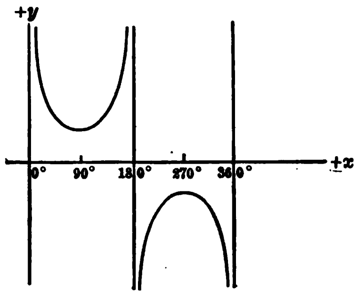

FIG. 97. — The Cosecant Curve

74. The natural unit of circular measurement. Definition of a radian. In constructing the graphs of the trigonometric functions, the student may have observed that the units of measurement on the x- and y-axes were *both* chosen arbitrarily, and might have been selected in infinitely many different ways, thus altering materially the appearance of the resulting curve.

This mutual independence of the two scales, on the two coördinate axes, is a natural consequence of the fact that

the two quantities x and y were regarded as different in kind. One of them was regarded as an angle measured in degrees, and the other as an abstract number. Having chosen a certain horizontal line-segment as representative of the unit of angles ($1°$), it was still admissible to choose arbitrarily another (vertical) line-segment to represent the unit of abstract numbers. Whenever x and y represent two quantities of different kind, the x- and y-scales are, in the nature of things, independent of each other.

Even if x and y are quantities of the same kind, it is often more convenient to choose the lengths of the units different on the two scales. If this were not done, the resulting curve might fail utterly to serve the purposes for which it was intended.

Thus, when we draw a profile map of an extensive country (showing the elevations of various points in a certain vertical cross section), the vertical scale must be chosen much larger than the horizontal scale. Otherwise the differences of elevation, as depicted on the map, would become so small as to be unnoticeable.

Nevertheless it will usually be desirable to choose the horizontal and vertical units equal to each other whenever x and y may be regarded as quantities of the same kind, provided that the resulting curve does not thereby lose its usefulness as it would in the example just quoted.

Now the considerations of Art. 70 show that, from a certain point of view, the quantities x and y which occur in such an equation as

$$y = \sin x$$

may be regarded as quantities of the same kind.

In fact, we observed in Art. 70 that we might think of the number x as the measure of the *arc* AP (Fig. 82) instead of as the measure of the corresponding *angle* AOP. We saw further that certain line-segments could be constructed whose lengths, in terms of the radius of the circle as unit, were equal to the values of $\sin x$, $\cos x$, etc. If then we measure the length of the arc x in terms of the radius of the circle as unit, instead of in degrees, we shall have the

arc and its trigonometric functions expressed in terms of the same unit.

This unit of arc measure, an arc of a circle whose length is equal to the radius of the circle, is called a **radian.**

One advantage gained by measuring arcs in radians is this: the arc and its trigonometric functions will then be expressed in terms of the same unit.

If r is the radius of a circle, the length of its circumference is equal to $2\pi r$. Therefore a circumference may be said to contain 2π radians. Since it also contains 360 degrees, we have

$$(1) \qquad 2\pi \text{ radians} = 360^\circ,$$

whence

$$1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi}.$$

Since $\pi = 3.14159265$, we find, to seven decimal places,

$$(2) \qquad 1 \text{ radian} = 57^\circ.2957795.$$

On the other hand we find from (1)

$$(3) \qquad 1^\circ = \frac{\pi}{180} \text{ radians},$$

or

$$(4) \qquad 1^\circ = 0.0174533 \text{ radian}.$$

These equations make it easy to find the number of degrees in an angle or arc when its measure is given in radians, or *vice versa.*

Clearly it follows, from the definition of a radian, that the length of the arc which an angle of one radian intercepts on the circumference of a circle of radius r is itself equal to r. Then, an angle of half a radian at the center will intercept an arc on the circumference whose length is equal to $\frac{1}{2}r$.

In general: *an angle of θ radians at the center of a circle of radius r intercepts an arc s upon the circumference whose length is*

$$(5) \qquad s = r\theta.$$

In the simplicity of this formula lies a second great advantage of measuring angles in radians rather than in degrees.

EXERCISE XLII

Convert into radians the following angles:

1. $90°$. 3. $30°$. 5. $+693° 20'$.

2. $270°$. 4. $+25° 15'$. 6. $-1030° 0'$.

Convert into degrees the following angles which are given in radians, state their quadrants and the signs of their trigonometric functions:

7. $\frac{\pi}{2}$. 9. $\frac{\pi}{16}$. 11. 0.7691.

8. $\frac{\pi}{4}$. 10. 3.14159. 12. 5.3214.

13. Prove that the area of a sector of a circle of radius r is equal to $\frac{1}{2}r^2\theta$ if θ, the angle at the center, is measured in radians.

14. Prove that a segment of a circle of radius r, whose arc is equal to θ radians, has the area

$$\tfrac{1}{2}r^2(\theta - \sin\theta).$$

15. Compute the area of a circular segment of radius 11 feet, if its arc is equal to 52°.

16. How will the formulæ of Exs. 13 and 14 be modified, if the angle θ is expressed in degrees?

17. A cord is stretched around two wheels, a large one of radius r and a smaller one of radius r' feet, the distance between the centers of the wheels being d feet. If the cord is not crossed and if θ is the angle of inclination, expressed in radians, of the free part of the cord to the line of centers of the wheels, show that

$$\sin\theta = \frac{r - r'}{d}, \quad l = 2\left[d\cos\theta + \left(\frac{\pi}{2} + \theta\right)r + \left(\frac{\pi}{2} - \theta\right)r'\right],$$

where l is the entire length of the cord.

18. If the cord in Ex. 17 is crossed, show that its length l may be found by means of the formulæ

$$\sin\theta = \frac{r + r'}{d}, \; l = 2\left[d\cos\theta + \left(\frac{\pi}{2} + \theta\right)(r + r')\right].$$

19. Find the length of a belt which is to be stretched around two wheels 3 and 2 feet in diameter respectively, if the distance between the centers of the two wheels is 5 feet: (*a*) if the belt is crossed, (*b*) if it is not crossed.

20. Draw the graphs of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, if x is expressed in radians, using the same unit of length for x distances and y distances.

75. Relations between the functions of two symmetrical angles. Let us consider two angles, like $90° - \theta$ and $90° + \theta$, which differ from one of the cardinal angles by the same amount but in opposite directions. If we place two such angles in the standard position, their terminal sides will be symmetrically situated with respect to one of the two coördinate axes, so that this axis will bisect the angle between them. As a consequence of this fact the trigonometric functions of the two angles are related to each other in a very simple fashion.

In order to obtain these relations we shall consider each of the four cardinal angles separately, making use of the line representation of the functions given in Art. 70.

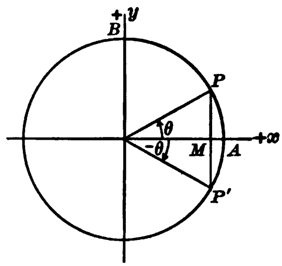

FIG. 98

Let us begin with the cardinal angle 0° or 0 radians. Figure 98 represents the unit circle and the two angles

$$\theta = \angle AOP \text{ and } -\theta = \angle AOP',$$

each in its standard position. The arcs AP and AP' subtended by these angles on the unit circle are symmetrical with respect to A. Therefore the ordinates of P and P' (the termini of these arcs) are numerically equal and opposite in sign, while their abscissas are equal in magnitude and sign.

But the ordinate and abscissa of the terminus of the arc AP are respectively equal to the sine and cosine of θ. The ordinate and abscissa of P' are respectively equal to the sine and cosine of $-\theta$. (See Art. 70.)

Consequently we have

$$(1) \qquad \sin(-\theta) = -\sin\theta, \ \cos(-\theta) = \cos\theta.$$

Consider next the case of two angles $90° - \theta$ and $90° + \theta$ symmetric with respect to the cardinal angle 90°. In this case (see Fig. 99) P and P' have the same ordinate, while their abscissas are numerically equal but opposite in sign. Consequently

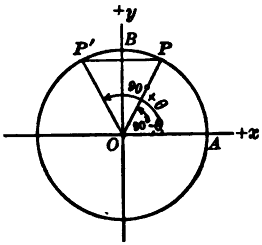

FIG. 99

$$(2\,a)\qquad \begin{aligned} \sin(90° - \theta) &= \sin(90° + \theta), \\ \cos(90° - \theta) &= -\cos(90° + \theta), \end{aligned}$$

or, if the angles are measured in radians,

$$(2\,b)\qquad \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right),\ \cos\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2} + \theta\right).$$

By a precisely similar argument, the details of which we leave to the student, we find

$$(3\,a)\qquad \begin{aligned} \sin(180° - \theta) &= -\sin(180° + \theta), \\ \cos(180° - \theta) &= \cos(180° + \theta); \end{aligned}$$

or,

$$(3\,b)\qquad \begin{aligned} \sin(\pi - \theta) &= -\sin(\pi + \theta), \\ \cos(\pi - \theta) &= \cos(\pi + \theta), \end{aligned}$$

according as the angle is measured in degrees or radians, and also

$$(4\,a)\qquad \begin{aligned} \sin(270° - \theta) &= \sin(270° + \theta), \\ \cos(270° - \theta) &= -\cos(270° + \theta); \end{aligned}$$

or,

$$(4\,b)\qquad \begin{aligned} \sin\left(\frac{3\pi}{2} - \theta\right) &= \sin\left(\frac{3\pi}{2} + \theta\right), \\ \cos\left(\frac{3\pi}{2} - \theta\right) &= -\cos\left(\frac{3\pi}{2} + \theta\right). \end{aligned}$$

Of course we shall also have

$$\begin{aligned} \sin(360° - \theta) &= -\sin(360° + \theta), \\ \cos(360° - \theta) &= \cos(360° + \theta). \end{aligned}$$

But these equations are really repetitions of (1) if we remember that, on account of the periodicity of the sine and cosine,

$$\sin(360° + \theta) = \sin\theta,\ \sin(360° - \theta) = \sin(-\theta),$$
$$\cos(360° + \theta) = \cos\theta,\ \cos(360° - \theta) = \cos(-\theta).$$

The relations, which correspond to (1), (2), (3), (4) for the remaining trigonometric functions, may easily be obtained by expressing the tangent, cotangent, secant, and cosecant in terms of the sine and cosine. (See Art. 68.) Thus, for instance,

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

The student should actually work out the sixteen equations obtainable in this way and combine them in tabular form with the eight equations (1) to (4).

The student should also observe that, although we have constructed the figures for the case when θ is a positive acute angle, our proof of formulæ (1), (2), (3), (4) remains valid word for word, if θ is a positive or negative angle of any magnitude. A good way to convince one's self of this fact is to think of the angle θ as variable and to follow out mentally the changes which would take place in such a figure as Fig. 98 or Fig. 99 when the angle θ increases or decreases. The fact that equations (1) to (4) are universally valid will thus be rendered intuitive.

76. Relations between functions of two angles whose sum or difference is a right angle. If θ is an acute angle, we know, from Art. 10, that

(1) $\sin(90° - \theta) = \cos\theta,\ \cos(90° - \theta) = \sin\theta.$

If we combine these equations with (2 a) of Art. 75, we find further

(2) $\sin(90° + \theta) = \cos\theta,\ \cos(90° + \theta) = -\sin\theta.$

We wish to show that the four equations (1) and (2) are true, not merely when θ is an acute angle, but when θ is a

positive or negative angle of any magnitude. This may be done by the *method of mathematical induction.*

We begin by proving the following theorem. *If equations (1) and (2) are true for a certain angle θ, they are also true for the angle $\theta' = 90° + \theta$.*

PROOF. By hypothesis, equations (1) and (2) are true for the angle θ. Therefore we have

$$\sin\theta' = \sin(90° + \theta) = \cos\theta,\quad \cos\theta' = \cos(90° + \theta) = -\sin\theta.$$

But

$$\sin(90° - \theta') = \sin[90° - (90° + \theta)] = \sin(-\theta) = -\sin\theta,$$

and

$$\cos(90° - \theta') = \cos[90° - (90° + \theta)] = \cos(-\theta) = \cos\theta,$$

(Art. 75, equations (1)),

which proves that

$$(3)\qquad \sin(90° - \theta') = \cos\theta',\quad \cos(90° - \theta') = \sin\theta',$$

since both members of the first equation are equal to $-\sin\theta$, and both members of the second are equal to $\cos\theta$.

Since equations (2 *a*) of Art. 75 are true for all angles, we now find

$$(4)\qquad \begin{aligned} \sin(90° + \theta') &= \sin(90° - \theta') = \cos\theta', \\ \cos(90° + \theta') &= -\cos(90° - \theta') = -\sin\theta'. \end{aligned}$$

Since equations (3) and (4) are the same as (1) and (2), with θ' in place of θ, we have actually proved our theorem; namely, *if* equations (1) and (2) are true for the angle θ, they are also true for the angle $\theta' = 90° + \theta$.

We know that equations (1) and (2) *are* true for all positive acute angles. As a consequence of the theorem just proved, they are successively seen to be true for all positive angles in the second, third, or fourth quadrant, and consequently for all positive angles whatever.

But they are also true for all negative angles. For let θ be a negative angle. Let $n \cdot 360°$, where n is a positive integer, be the lowest integral multiple of 360° which makes

$$\theta' = \theta + n \cdot 360°$$

a positive angle. Then, on account of the periodic character of the sine and cosine, we shall have

$$(5)\qquad \begin{aligned}\sin\theta' &= \sin(\theta + n\cdot 360^\circ) = \sin\theta,\\ \cos\theta' &= \cos(\theta + n\cdot 360^\circ) = \cos\theta,\end{aligned}$$

and similarly

$$(6)\qquad \begin{aligned}&\sin(90^\circ-\theta')=\sin(90^\circ-\theta),\ \cos(90^\circ-\theta')=\cos(90^\circ-\theta),\\ &\sin(90^\circ+\theta')=\sin(90^\circ+\theta),\ \cos(90^\circ+\theta')=\cos(90^\circ+\theta).\end{aligned}$$

Since θ' is a positive angle, we have

$$\begin{aligned}&\sin(90^\circ-\theta')=\cos\theta',\ \cos(90^\circ-\theta')=\sin\theta',\\ &\sin(90^\circ+\theta')=\cos\theta',\ \cos(90^\circ+\theta')=-\sin\theta'.\end{aligned}$$

If in these equations we substitute the values (5) and (6), we find

$$\begin{aligned}&\sin(90^\circ-\theta)=\cos\theta,\ \cos(90^\circ-\theta)=\sin\theta,\\ &\sin(90^\circ+\theta)=\cos\theta,\ \cos(90^\circ+\theta)=-\sin\theta.\end{aligned}$$

Therefore *equations* (1) *and* (2) *are true for positive and negative angles of any magnitude.*

The formulæ for $\tan(90^\circ+\theta)$, $\sec(90^\circ+\theta)$, etc., may be found by expressing these functions of $90^\circ+\theta$ in terms of $\sin(90^\circ+\theta)$ and $\cos(90^\circ+\theta)$ and making use of (2); for instance, we find

$$\tan(90^\circ+\theta)=\frac{\sin(90^\circ+\theta)}{\cos(90^\circ+\theta)}=\frac{\cos\theta}{-\sin\theta}=-\cot\theta.$$

77. The quadrantal formulæ. If we unite equations (1) of Art. 76 with the corresponding formulæ for the remaining four functions, we obtain the following system of equations:

$$(1)\qquad \begin{cases}\sin(90^\circ-\theta)=\cos\theta, & \cos(90^\circ-\theta)=\sin\theta,\\ \tan(90^\circ-\theta)=\cot\theta, & \cot(90^\circ-\theta)=\tan\theta,\\ \sec(90^\circ-\theta)=\csc\theta, & \csc(90^\circ-\theta)=\sec\theta.\end{cases}$$

In the same way we find, from equations (2) of Art. 76, the system:

$$(2)\qquad \begin{cases}\sin(90^\circ+\theta)=\cos\theta, & \cos(90^\circ+\theta)=-\sin\theta,\\ \tan(90^\circ+\theta)=-\cot\theta, & \cot(90^\circ+\theta)=-\tan\theta,\\ \sec(90^\circ+\theta)=-\csc\theta, & \csc(90^\circ+\theta)=+\sec\theta.\end{cases}$$

Since these equations are true for angles of any magnitude and not merely for acute angles, we conclude from (1) and (2) that

$$\sin(180° - \theta) = \sin(90° + \overline{90° - \theta}) = \cos(90° - \theta) = \sin\theta,$$
$$\cos(180° - \theta) = \cos(90° + \overline{90° - \theta}) = -\sin(90° - \theta) = -\cos\theta,$$

etc., giving rise to the further system of equations:

$$(3)\quad \begin{cases} \sin(180° - \theta) = \sin\theta, & \cos(180° - \theta) = -\cos\theta, \\ \tan(180° - \theta) = -\tan\theta, & \cot(180° - \theta) = -\cot\theta, \\ \sec(180° - \theta) = -\sec\theta, & \csc(180° - \theta) = \csc\theta. \end{cases}$$

In similar fashion we find

$$\sin(180° + \theta) = \sin(90° + \overline{90° + \theta}) = \cos(90° + \theta) = -\sin\theta,$$
$$\cos(180° + \theta) = \cos(90° + \overline{90° + \theta}) = -\sin(90° + \theta)$$
$$= -\cos\theta, \text{ etc.,}$$

so that we obtain

$$(4)\quad \begin{cases} \sin(180° + \theta) = -\sin\theta, & \cos(180° + \theta) = -\cos\theta, \\ \tan(180° + \theta) = \tan\theta, & \cot(180° + \theta) = \cot\theta, \\ \sec(180° + \theta) = -\sec\theta, & \csc(180° + \theta) = -\csc\theta. \end{cases}$$

Again we have

$$\sin(270° - \theta) = \sin(90° + \overline{180° - \theta}) = \cos(180° - \theta)$$
$$= -\cos\theta, \text{ etc.,}$$

whence

$$(5)\quad \begin{cases} \sin(270° - \theta) = -\cos\theta, & \cos(270° - \theta) = -\sin\theta, \\ \tan(270° - \theta) = \cot\theta, & \cot(270° - \theta) = \tan\theta, \\ \sec(270° - \theta) = -\csc\theta, & \csc(270° - \theta) = -\sec\theta, \end{cases}$$

and similarly

$$(6)\quad \begin{cases} \sin(270° + \theta) = -\cos\theta, & \cos(270° + \theta) = \sin\theta, \\ \tan(270° + \theta) = -\cot\theta, & \cot(270° + \theta) = -\tan\theta, \\ \sec(270° + \theta) = \csc\theta, & \csc(270° + \theta) = -\sec\theta. \end{cases}$$

Finally we find the equations

$$(7)\quad \begin{cases} \sin(360° - \theta) = -\sin\theta, & \cos(360° - \theta) = \cos\theta, \\ \tan(360° - \theta) = -\tan\theta, & \cot(360° - \theta) = -\cot\theta, \\ \sec(360° - \theta) = \sec\theta, & \csc(360° - \theta) = -\csc\theta, \end{cases}$$

and the system

$$(8)\quad \begin{cases} \sin(360° + \theta) = \sin\theta, & \cos(360° + \theta) = \cos\theta, \\ \tan(360° + \theta) = \tan\theta, & \cot(360° + \theta) = \cot\theta, \\ \sec(360° + \theta) = \sec\theta, & \csc(360° + \theta) = \csc\theta, \end{cases}$$

which latter equations merely express the periodic character of the trigonometric functions.

Since, on account of the periodicity of the functions, we have

$$\sin(360° - \theta) = \sin(-\theta + 360°) = \sin(-\theta), \text{ etc.,}$$

we may also write, in place of (7),

$$(9)\quad \begin{cases} \sin(-\theta) = -\sin\theta, & \cos(-\theta) = \cos\theta, \\ \tan(-\theta) = -\tan\theta, & \cot(-\theta) = -\cot\theta, \\ \sec(-\theta) = \sec\theta, & \csc(-\theta) = -\csc\theta. \end{cases}$$

The 48 formulæ (1) to (8), the so-called **quadrantal formulæ**, have a very important practical application. *They serve the purpose of finding the values of the functions of angles not situated in the first quadrant.*

For instance, if we wish to find the sine and cosine of 310°, we may use equations (6), which give

$$\sin 310° = \sin(270° + 40°) = -\cos 40°,$$
$$\cos 310° = \cos(270° + 40°) = \sin 40°.$$

The numerical values of cos 40° and sin 40° may, of course, be taken from the tables.

On account of the practical application just mentioned, it is important to be able to remember the 48 quadrantal formulæ. This is not at all difficult if we impress upon our minds some of their peculiarities.

We observe in the first place that all of the angles which appear in the left members of these equations are of the form

a multiple of $90° \pm \theta$, that is, *a cardinal angle* $\pm\ \theta$,

while the angle which appears in the right member is always simply θ.

Let us speak of those cardinal angles, like 90° and 270°, which are odd multiples of 90° as *odd cardinal angles*, while

the *even cardinal angles*, like 0° and 180°, are even multiples of 90°.

If now we look through our list of quadrantal formulæ, we observe that they are all included in one of the two forms:

$$(10)\quad \begin{cases} \textit{function of } (\textbf{even} \textit{ cardinal angle } \pm \theta) \\ \qquad\qquad = \pm\, \textbf{same} \textit{ function of } \theta, \\ \textit{function of } (\textbf{odd} \textit{ cardinal angle } \pm \theta) \\ \qquad\qquad = \pm\, \textit{corresponding } \textbf{co-}\textit{function of } \theta, \end{cases}$$

it being understood, as in Art. 10, that the six functions are arranged in three pairs, sine and cosine, tangent and cotangent, secant and cosecant, each member of each pair being regarded as the co-function of the other.

We have observed, then, that *the same function occurs in both members of a quadrantal formula whenever the corresponding cardinal angle is even. If the cardinal angle is odd, the function which appears in the right member is the co-function of that one which appears on the left.*

It remains to describe a method for remembering which of the two signs, + or −, should be used in any one of these formulæ. We may determine this sign by thinking of the particular case when θ is a positive acute angle. The quadrant of the angle in the left member of the equation will then be evident by inspection, and therefore also the sign of the left member. The ambiguous sign ±, on the right member, must then be chosen in such a manner as to make the two members of the equation agree in sign.

An example will make this clear. We wish to find the formula for $\sin(270° + \theta)$. Since 270° is an odd cardinal angle, we have in the first place

$$\sin(270° + \theta) = \pm \cos\theta.$$

To determine the sign on the right member, we think of the case when θ is a positive acute angle. Then $270°+\theta$ is in the fourth quadrant, and $\sin(270°+\theta)$ is negative, while $\cos\theta$, being the cosine of a positive acute angle, is positive. Therefore we must choose the − sign, so that

$$\sin(270° + \theta) = -\cos\theta,$$

since choice of the + sign would lead to the absurdity of equating a positive to a negative number.

EXERCISE XLIII

Express the following as functions of positive acute angles:

1. $\cos(-75°)$.
2. $\sin 325°$.
3. $\tan 517°$.
4. $\csc(-412°)$.
5. $\cot 175°$.
6. $\sec 1562°$.

7. Express the functions of Exs. 1–6 as functions of positive acute angles less than 45°.

8. Compute the value of the expression $2\sin(3\theta + 10°)$ for the values of $\theta = 25°, 50°, 75°, 100°$.

Find the values of the following expressions:

9. $\cos 60° \cos 120° - \sin 60° \sin 120°$.

10. $\sin 30° \cos 300° + \cos 30° \sin 300°$.

11. $\tan \frac{17\pi}{6} \tan \frac{14\pi}{3} + \cot\left(\frac{-11\pi}{6}\right) \cot\left(\frac{-4\pi}{3}\right)$.

12. $\cos 315° \sin 11° - \tan 293° \sec 25°$.

Simplify the following expressions:

13. $a^2 + b^2 + 2ab\cos(180° - x)$.

14. $(a - b)\tan(90° + A) + (a + b)\cot(-A)$.

15. $a\sin\left(\frac{3\pi}{2} - \theta\right) + b\cos(\pi - \theta)$.

16. $\tan\theta + \tan(\pi - \theta)$.

State for what values of θ each of the following expressions is positive, and for what values of θ it is negative:

17. $\sin\theta - \cos\theta$.
18. $\sin\theta + \cos\theta$.
19. $\sin^2\theta - \cos^2\theta$.
20. $\tan\theta - \cot\theta$.

21. Find the formulæ for the functions of $\theta - \frac{\pi}{2}$ in terms of the functions of θ, the angle θ being measured in radians.

22. Find formulæ for the functions of $\theta - \pi$ in terms of the functions of θ.

78. Properties of the sine and cosine curves. The properties of the sine and cosine which were discussed in Arts. 75, 76, 77 manifest themselves very clearly if we make use of the graphs of these functions as obtained in Art. 73. We shall slightly modify these graphs, however, by thinking of

the angle x as being measured in radians (see Art. 74) rather than in degrees, and by choosing the same unit of length to

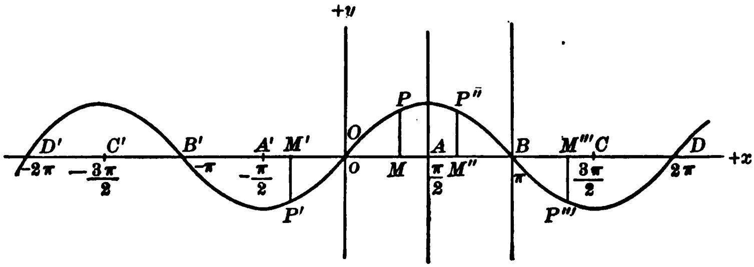

FIG. 100. — The Sine Curve. Natural Scale

represent one radian on the x-axis as that which represents the abstract number 1 on the y-axis.

Figure 100 represents the sine curve

$$y = \sin x$$

constructed in accordance with this choice of units.

Let us consider two points of this curve which are at the same distance from the y-axis but on opposite sides, such as P and P'. The ordinates of these points are obviously numerically equal but opposite in sign. Denote the abscissa of P by z; then, that of P' will be $-z$, and we find that

(1). $$\sin(-z) = -\sin z.$$

If we draw a line parallel to the y-axis through the point A for which $x = \frac{\pi}{2}$, this line clearly divides the curve into two symmetrical portions. Consequently two points, such as P and P'', at the same distance from this line but on opposite sides of it, will have equal ordinates. That is,

$$\sin OM = \sin OM''.$$

But if we denote MA by z, we have

$$OM = \frac{\pi}{2} - z, \quad OM'' = \frac{\pi}{2} + z,$$

so that

(2). $$\sin\left(\frac{\pi}{2} - z\right) = \sin\left(\frac{\pi}{2} + z\right).$$

Let the curve be divided into two portions by a line parallel to OY through the point B whose abscissa is equal to π. Points of the curve at equal distances from this line but on opposite sides of it, such as P'' and P''', have ordinates numerically equal but opposite in sign. Therefore

$$(3)_s \qquad \sin(\pi - z) = -\sin(\pi + z).$$

In similar fashion we find

$$(4)_s \qquad \sin\left(\frac{3\pi}{2} - z\right) = \sin\left(\frac{3\pi}{2} + z\right),$$

$$(5)_s \qquad \sin(2\pi - z) = -\sin(2\pi + z).$$

The points M and M'' were equidistant from A. Consequently the distances OM and $M''B$ are equal. Since the ordinates MP and $M''P''$ are equal, we shall have

$$\sin OM'' = \sin OM.$$

If we put $OM = z$, we shall have $M''B = z$, and therefore

$$OM'' = \pi - z,$$

so that the preceding equation becomes

$$(6)_s \qquad \sin(\pi - z) = \sin z.$$

By similar considerations in connection with the cosine curve, we find a system of relations which correspond completely to the above equations $(1)_s$ to $(6)_s$. They are

$$(1)_c \qquad \cos(-z) = \cos z,$$

$$(2)_c \qquad \cos\left(\frac{\pi}{2} - z\right) = -\cos\left(\frac{\pi}{2} + z\right),$$

$$(3)_c \qquad \cos(\pi - z) = \cos(\pi + z),$$

$$(4)_c \qquad \cos\left(\frac{3\pi}{2} - z\right) = -\cos\left(\frac{3\pi}{2} + z\right),$$

$$(5)_c \qquad \cos(2\pi - z) = \cos(2\pi + z),$$

$$(6)_c \qquad \cos(\pi - z) = -\cos z.$$

The truth of all these equations which are thus suggested by the curves has been established, in a slightly different notation, in Arts. 75–77.

We can hardly fail to notice the striking similarity between the sine and cosine curves. In order to put into evidence the relations between them, we construct Fig. 101 which contains them both.

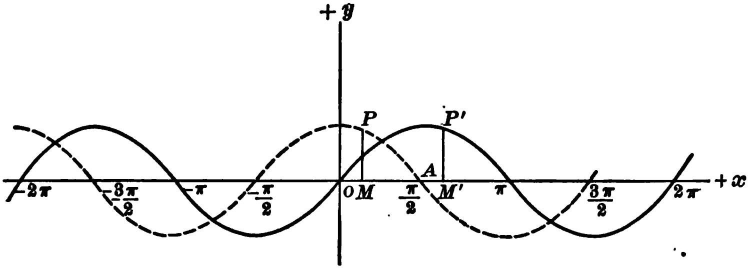

FIG. 101

This figure suggests that, if we displace the cosine curve toward the right through a distance of $\frac{\pi}{2}$ units, it will coincide with the sine curve. If this is true, a point P of the cosine curve whose coördinates are

$$(a) \qquad OM = z, \quad MP = \cos z,$$

and which by our displacement will be brought into coincidence with a point P' whose coördinates are

$$(b) \qquad OM' = z + \frac{\pi}{2}, \quad M'P' = MP,$$

should, in its new position, be a point on the sine curve. Therefore the coördinates, OM' and $M'P'$, of this point P' should satisfy the relation $y = \sin x$ which is satisfied by the coördinates of all points of the sine curve. This gives

$$(c) \qquad M'P' = \sin\left(z + \frac{\pi}{2}\right),$$

and, on combination with (a) and (b),

$$(7). \qquad \sin\left(z + \frac{\pi}{2}\right) = M'P' = MP = \cos z.$$

Thus, equation (7), must be true if the geometric relation between the sine and cosine curves suggested by Fig. 101 is actually based on fact. But this equation coincides with formula (2) of Art. 76, except for the notation, so that its validity is no longer open to question.

Consequently, *the sine and cosine curve differ only in position and may be brought into coincidence by a displacement of* $\frac{\pi}{2}$ *units parallel to the x-axis.*

EXERCISE XLIV

1. What geometric property of Fig. 101 corresponds to the relations

$$\sin\left(\frac{\pi}{2}-z\right)=\cos z,\ \cos\left(\frac{\pi}{2}-z\right)=\sin z?$$

2. How are the relations

$$\sin\left(\frac{3\pi}{2}+z\right)=-\cos z,\ \cos\left(\frac{3\pi}{2}+z\right)=\sin z$$

to be obtained from Fig. 101?

3. Plot the tangent and cotangent curves and discuss these graphs in a fashion analogous (so far as possible) to the discussion of Art. 78.

CHAPTER XI

RELATIONS BETWEEN THE FUNCTIONS OF MORE THAN ONE ANGLE

79. The addition theorems for sine and cosine. In Art. 77 we expressed $\sin\left(\theta + \frac{\pi}{2}\right)$, $\cos\left(\theta + \frac{\pi}{2}\right)$, etc., in terms of $\sin\theta$ and $\cos\theta$. The angle θ was an angle of any magnitude, but the angle added to it was always an integral multiple of $\frac{\pi}{2}$ radians or 90°. The question now arises whether it is possible to find similar formulæ for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, where both α and β are angles of any magnitude.

Let us assume, to begin with, that α and β are both positive acute angles whose sum $\alpha + \beta$ is also acute. We place the acute angle α in its standard position xOA. (See Fig. 102.) We then place the angle β with its initial side upon OA (the terminal side of the angle α), so as to make $\angle AOB$ equal to β. Then

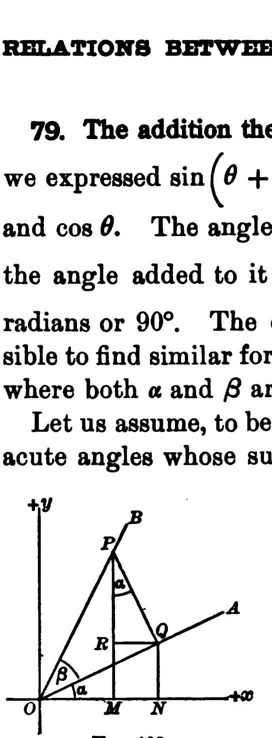

FIG. 102

$$\angle xOB = \alpha + \beta,$$

and moreover this angle is in its standard position. Therefore if we take any point P, different from O, on its terminal side OB and drop a perpendicular PM from P to the x-axis, we shall have

$$(1) \qquad \sin(\alpha + \beta) = \frac{MP}{OP}, \quad \cos(\alpha + \beta) = \frac{OM}{OP}.$$

Let us drop perpendiculars PQ, QN, and QR from P to OA, from Q to the x-axis, and from Q to MP. Then we have

$$(2) \qquad \sin(\alpha + \beta) = \frac{MP}{OP} = \frac{NQ + RP}{OP} = \frac{NQ}{OP} + \frac{RP}{OP}.$$

Now $\frac{NQ}{OP}$ is a ratio of sides of two *different right triangles*, namely, ONQ and OPQ. But these triangles have the side OQ in common, and this common side may be used to transform $\frac{NQ}{OP}$ into a product of two ratios, each of which contains two sides of the *same* right triangle and is therefore a trigonometric function of the acute angles of this triangle. In fact we find

$$(3) \qquad \frac{NQ}{OP} = \frac{NQ}{OQ} \cdot \frac{OQ}{OP} = \sin\alpha\cos\beta.$$

In the same way we find for the second term of (2)

$$\frac{RP}{OP} = \frac{RP}{PQ} \cdot \frac{PQ}{OP}.$$

But

$$\frac{PQ}{OP} = \sin\beta, \text{ and } \frac{RP}{PQ} = \cos RPQ = \cos\alpha,$$

since the sides of the angle RPQ are respectively perpendicular to those of α. Consequently we find

$$(4) \qquad \frac{RP}{OP} = \cos\alpha\sin\beta.$$

If (3) and (4) be substituted in (2), we obtain the important formula

$$(5) \qquad \sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Referring once more to Fig. 102, we have

$$(6) \qquad \cos(\alpha + \beta) = \frac{OM}{OP} = \frac{ON - RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}.$$

If we again transform each of these ratios into a product of two others, we find

$$\frac{ON}{OP} = \frac{ON}{OQ} \cdot \frac{OQ}{OP} = \cos\alpha\cos\beta,$$

$$\frac{RQ}{OP} = \frac{RQ}{PQ} \cdot \frac{PQ}{OP} = \sin\alpha\sin\beta,$$

so that

(7) $$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta.$$

It is easy to see that equations (5) and (7) remain valid if α and β are acute angles, even if their sum is greater than a right angle. In that case we shall have the situation represented in Fig. 103. If we make precisely the same constructions as before, the proof of the formula for $\sin(\alpha + \beta)$ will remain applicable word for word. But since $\alpha + \beta$ is now in the second quadrant, its cosine is negative; that is,

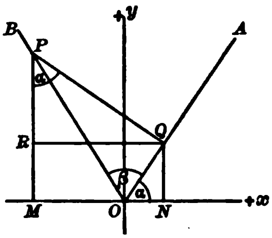

FIG. 103

$$\cos(\alpha + \beta) = -\frac{MO}{OP},$$

where by MO we mean merely the positive number expressing the length of the line-segment MO, so that $-MO$ is a negative number.

Now the figure shows that

$$MO = MN - ON = RQ - ON,$$

so that

$$\cos(\alpha + \beta) = -\frac{MO}{OP} = \frac{ON - RQ}{OP} = \frac{ON}{OP} - \frac{RQ}{OP}.$$

From this point on, the proof proceeds exactly as in the previous case, beginning from equation (6).

Thus, we have proved that equations (5) and (7) are certainly true if α and β are positive acute angles, even if their sum is greater than 90°.

We may now show that these formulæ are true for two angles in any quadrant. In order to do this, we first prove the following theorem. *If the formulæ*

(8) $$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta,$$
$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta,$$

are true for two angles α and β, they remain true if either of these angles be increased by 90°.

PROOF. Let us assume that equations (8) are true for a certain pair of angles α and β. Put

$$\alpha' = 90° + \alpha,$$

so that

$$\alpha' + \beta = 90° + \alpha + \beta.$$

We shall then have (Art. 77, equations (2)),

$$(9)\quad \begin{aligned} \sin(\alpha' + \beta) &= \sin(90° + \alpha + \beta) = \cos(\alpha + \beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\ \cos(\alpha' + \beta) &= \cos(90° + \alpha + \beta) = -\sin(\alpha + \beta) \\ &= -\sin\alpha\cos\beta - \cos\alpha\sin\beta, \end{aligned}$$

and also

$$\sin\alpha' = \sin(90° + \alpha) = \cos\alpha,\quad \cos\alpha' = \cos(90° + \alpha) = -\sin\alpha,$$

whence

$$\sin\alpha = -\cos\alpha',\qquad \cos\alpha = \sin\alpha'.$$

If we substitute these values in (9), we find

$$\sin(\alpha' + \beta) = \sin\alpha'\cos\beta + \cos\alpha'\sin\beta,$$
$$\cos(\alpha' + \beta) = \cos\alpha'\cos\beta - \sin\alpha'\sin\beta.$$

But these formulæ are of the same form as equations (8), with $\alpha' = \alpha + 90°$ in place of α. The same process would show that equations (8) would still be satisfied if we replaced β by $\beta' = 90° + \beta$. Consequently our theorem is proved.

We know already that equations (8) *are* true if α and β are any two positive acute angles. On account of the theorem just proved they will still be true if α is in the second quadrant and β in the first, and hence also if both α and β are in the second quadrant, and hence if one of these angles is in the third quadrant, etc. Therefore equations (8) are true for all positive angles.

But they are also true if either or both angles are negative. For instance, let α be a negative angle while β is positive. By adding to α a sufficient number n of complete positive revolutions, we shall obtain

$$\alpha' = \alpha + n \cdot 360°,$$

a positive angle. But then

$$\sin \alpha' = \sin \alpha, \qquad \cos \alpha' = \cos \alpha,$$

$$\sin(\alpha' + \beta) = \sin(\alpha + \beta), \quad \cos(\alpha' + \beta) = \cos(\alpha + \beta),$$

so that

$$\begin{aligned} \sin(\alpha + \beta) = \sin(\alpha' + \beta) &= \sin \alpha' \cos \beta + \cos \alpha' \sin \beta \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta, \end{aligned}$$

and similarly for $\cos(\alpha + \beta)$.

If β is negative, we may proceed in the same manner. Consequently we obtain the following important theorem:

If α and β are two positive or negative angles of any magnitude, the sine and cosine of their sum are always given by the formulæ

$$(8) \qquad \begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta, \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned}$$

The *method of mathematical induction* employed for proving the general validity of these formulæ is of fundamental importance in all parts of mathematics. We have already made use of it in Art. 76.

Equations (8) are usually known as the *addition theorems* for sine and cosine.

EXERCISE XLV

1. Compute the sine and cosine of 75°.

Solution. We have $75° = 30° + 45°$, $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{1}{2}\sqrt{3}$, $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$. Therefore

$$\sin 75° = \sin 30° \cos 45° + \cos 30° \sin 45° = ?$$
$$\cos 75° = \cos 30° \cos 45° - \sin 30° \sin 45° = ?$$

Compute the sines and cosines of the following angles:

2. 120°. **4.** 210°. **6.** 195°.

3. 150°. **5.** 105°. **7.** 165°.

8. Find formulæ for $\sin(45° + \theta)$ and $\cos(45° + \theta)$.

9. Find formulæ for $\sin(30° + \theta)$ and $\cos(30° + \theta)$.

10. Find formulæ for $\sin(60° + \theta)$ and $\cos(60° + \theta)$.

11. Show that those of the quadrantal formulæ of Art. 77, which involve the sine and cosine, are special cases of the addition theorems for the sine and cosine.

Prove that the following equations are true for all values of the angles which appear in them. That is, prove that they are *identities*.

12. $\sin(\alpha+\beta)\cos\beta+\cos(\alpha+\beta)\sin\beta=\sin(\alpha+2\beta)$.

13. $\cos(\alpha+\beta)\cos\beta-\sin(\alpha+\beta)\sin\beta=\cos(\alpha+2\beta)$.

14. Show how $\sin(\alpha+\beta+\gamma)$ and $\cos(\alpha+\beta+\gamma)$ may be expressed in terms of the functions of α, β, and γ.

Simplify the following expressions:

15. $\sin(1+n)\theta\cos(1-n)\theta+\cos(1+n)\theta\sin(1-n)\theta$.

16. $\cos(1+n)\theta\cos(1-n)\theta-\sin(1+n)\theta\sin(1-n)\theta$.

80. The addition theorems for tangent and cotangent. The addition formulæ for the tangent and cotangent may be obtained as consequences of those for the sine and cosine. We have

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}.$$

The fraction in the right member of this equation may be expressed in terms of $\tan\alpha$ and $\tan\beta$ by dividing both numerator and denominator by $\cos\alpha\cos\beta$. We obtain in this way

$$\tan(\alpha+\beta)=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}},$$

or, finally,

$$(1)\qquad \boldsymbol{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}.$$

By a similar process we find

$$(2)\qquad \cot(\alpha+\beta)=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}.$$

EXERCISE XLVI

Find the tangents and cotangents of the following angles:

1. 75°.
2. 105°.
3. 210°.
4. 150°.

Prove that the following equations are identities:

5. $\tan(45° + \theta) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$.
6. $\cot(45° + \theta) = \dfrac{\cot\theta - 1}{\cot\theta + 1}$.
7. Find formulæ for $\tan(30° + \theta)$ and $\cot(30° + \theta)$.
8. Find formulæ for $\tan(60° + \theta)$ and $\cot(60° + \theta)$.

81. The subtraction formulæ. The formulæ for $\sin(\alpha - \beta)$, $\cos(\alpha - \beta)$, etc., may be obtained from those for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, etc. For, since the addition formulæ are known to be true for all positive and negative values of α and β, we may write

$$\sin(\alpha - \beta) = \sin[\alpha + (-\beta)] = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta),$$

$$\cos(\alpha - \beta) = \cos[\alpha + (-\beta)] = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta).$$

But, on account of equations (9), Art. 77, we have

$$\sin(-\beta) = -\sin\beta, \quad \cos(-\beta) = \cos\beta,$$

so that we find

$$(1)\qquad \begin{aligned} \sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta, \\ \cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta. \end{aligned}$$

In the same way we find

$$(2)\qquad \tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta},$$

and

$$(3)\qquad \cot(\alpha - \beta) = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}.$$

Of course equations (2) and (3) may also be obtained from (1) by division.

EXERCISE XLVII

1. Let the student devise a geometric proof for equations (1), in the case when α and β are positive acute angles, α being the larger of the two.

Note. If Fig. 102 be described in *words*, but if the angle there denoted by β be instead regarded and constructed as a negative angle, the same description will apply to both figures and the same method of transforming the ratios which was used in Art. 79 will be effective in the present example.

82. Formulæ for converting products of trigonometric functions into sums, and vice versa. Before the invention of logarithms the calculations of products and quotients was a very laborious process. In those days, then, it was considered a great simplification if a formula whose numerical evaluation required multiplication could be transformed into another requiring only addition or subtraction.* The addition and subtraction formulæ will enable us to accomplish this for a product of two sines, of two cosines, or of a sine and cosine.

In fact, we have the two equations

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$$

from which we derive, by addition and subtraction,

$$(1)\qquad \begin{aligned}\sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\sin\alpha\cos\beta,\\ \sin(\alpha+\beta) - \sin(\alpha-\beta) &= 2\cos\alpha\sin\beta.\end{aligned}$$

In the same way, from

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

we find

$$(2)\qquad \begin{aligned}\cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos\alpha\cos\beta,\\ \cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta.\end{aligned}$$

From these last equations we find, by transposition and division by 2,

$$(3)\qquad \begin{aligned}\cos\alpha\cos\beta &= \tfrac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)],\\ \sin\alpha\sin\beta &= \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)],\end{aligned}$$

* This was the so-called *prosthaphæretic method.*

whereas, from the first equation of (1), we find

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]. \tag{4}$$

The result obtained from the second equation of (1) is the same as (4) except for an interchange of the letters α and β.

Equations (3) *and* (4) *are important in that they enable us to transform a product of two sines, two cosines, or of a sine and a cosine into a sum or difference.*

For many purposes this is very important even nowadays,* although not for the purposes of numerical calculation. In fact, at the present time, for numerical work we prefer formulæ which involve multiplication to those involving addition, because the former process is more easily performed by logarithms.

The above formulæ, slightly modified, may also be used for the purpose of converting sums and differences of sines and cosines into products. Let us put

$$\alpha+\beta=A, \qquad \alpha-\beta=B,$$

so that

$$\alpha=\tfrac{1}{2}(A+B), \qquad \beta=\tfrac{1}{2}(A-B).$$

Then, equations (1) and (2) yield the following four formulæ:

$$(5)\quad \begin{cases} \sin A+\sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B), \\ \sin A-\sin B = 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B), \\ \cos A+\cos B = 2\cos\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B), \\ \cos A-\cos B = -2\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+B). \end{cases}$$

We have seen the first two of these equations before, having derived them directly from a figure (in Art. 49) for the purpose of proving the law of tangents. Our proof, at that time, did not permit us to affirm that the formulæ were true for all angles A and B. That such is the case, however, has now been made evident, since the present proof was obtained without placing any restrictions on the values of the angles A and B.

* For instance, in harmonic analysis (see Arts. 111 and 112) and in the integral calculus.

EXERCISE XLVIII

Prove the following equations:

1. $\sin 3\theta + \sin\theta = 2\sin 2\theta\cos\theta$.

2. $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right)=2\sin\frac{\pi}{4}\cos x=\sqrt{2}\cos x$.

3. $\dfrac{\sin 6\alpha+\sin 4\alpha}{\cos 6\alpha+\cos 4\alpha}=\tan 5\alpha$.

4. $\dfrac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}=\dfrac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}}$.

5. $\cos\alpha-\cos 3\alpha=2\sin\alpha\sin 2\alpha$.

6. $\sin 3\alpha+\cos\alpha=\sin 3\alpha+\sin(90°-\alpha)=?$

7. By generalizing the process observed in Ex. 6, derive a formula for $\sin\alpha+\cos\beta$.

8. Derive a formula for $\sin\alpha-\cos\beta$.

Reduce the following products to sums or differences:

9. $\sin 4\alpha\cos 2\alpha$.

10. $\sin 6\theta\sin 4\theta$.

11. $\cos 2\beta\sin 4\beta$.

12. $\cos 2\theta\cos 8\theta$.

13. $\sin 5\alpha\cos 3\alpha$.

14. $\sin^2\theta\cos\theta$.

15. Making use of the formulæ (5) of Art. 82, show how to derive the law of tangents from the law of sines.

83. Functions of double angles. If we put $\beta=\alpha$ in equations (8) of Art. 79, we find

(1) $$\mathbf{\sin 2\alpha = 2\sin\alpha\cos\alpha,}$$

and

(2) $$\mathbf{\cos 2\alpha=\cos^2\alpha-\sin^2\alpha.}$$

On account of the relation

$$\sin^2\alpha+\cos^2\alpha=1,$$

the latter equation may also be written in either of the following two forms:

(3) $$\cos 2\alpha=1-2\sin^2\alpha,$$

or

(4) $$\cos 2\alpha=2\cos^2\alpha-1.$$

If we put $\beta = \alpha$ in equations (1) and (2) of Art. 80, we find

$$\text{(5)} \qquad \tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha},$$

$$\text{(6)} \qquad \cot 2\alpha = \frac{\cot^2 \alpha - 1}{2 \cot \alpha},$$

which equations may, of course, also be derived from (1) and (2) by division.

84. Functions of half angles. In Art. 83 we regard the functions of α as known, and we learn how to compute the functions of 2α. We shall now invert the problem by regarding as known the functions of 2α, the problem being to calculate the functions of α, the half angle. To put the character of the problem more clearly into evidence, we shall put

$$2\alpha = \theta, \qquad \alpha = \tfrac{1}{2}\theta,$$

which merely amounts to thinking of any angle θ as a double angle; namely, as double its half.

With this change of notation, equations (3) and (4) of Art. 83 become

$$\text{(1)} \qquad \cos \theta = 1 - 2 \sin^2 \frac{\theta}{2}, \qquad \cos \theta = 2 \cos^2 \frac{\theta}{2} - 1.$$

If we solve the first of these equations for $2 \sin^2 \frac{\theta}{2}$ and the second for $2 \cos^2 \frac{\theta}{2}$, we find

$$\text{(2)} \qquad 2 \sin^2 \frac{\theta}{2} = 1 - \cos \theta, \qquad 2 \cos^2 \frac{\theta}{2} = 1 + \cos \theta,$$

whence

$$\text{(3)} \qquad \sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \qquad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}.$$

The ambiguous signs on the right members are determined by the quadrant of the angle $\frac{\theta}{2}$. If θ is a positive angle not greater than $180°$, $\frac{\theta}{2}$ is in the first quadrant and the $+$ sign must be chosen in both of the equations (3).

From (3) we find, by division,

$$(4)\qquad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}},\qquad \cot\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}},$$

the appropriate sign being again determined by the quadrant of the angle $\frac{1}{2}\theta$.

According to (1), Art. 83, we have

$$(5)\qquad 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta.$$

Let us divide each member of the first equation of (2) by the corresponding member of (5). We find

$$(6)\qquad \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}.$$

By a similar process we obtain, from the second equation of (2),

$$(7)\qquad \cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}.$$

These last two formulæ might also have been derived from (4). But the proof we have given is preferable since it avoids the necessity of discussing the ambiguous sign, a discussion which would be necessary if we had followed the other method.

The equations for $\sin\frac{\theta}{2}$, $\cos\frac{\theta}{2}$, etc., are of very great importance, because *they may be used for the purpose of computing a table of trigonometric functions.* In fact, we have already shown how to calculate the values of the functions of 0°, 30°, 45°, 60°, and 90° (Art. 11). By means of the addition and subtraction formulæ (Arts. 79, 81) we are therefore in a position to find also the functions of 15° and 75°. We can now calculate the values of the functions of one half of 15° or 7°.5, of one half of 7°.5, or 3°.75, etc. By continuing this process of bisection and combining the results by means of the addition theorem, we may obviously compute the values of the functions for a set of angles between 0° and 90° as

close together as we please. By interpolation we may then find the functions of 1°, 2°, 3°, etc.

The method, of which we have just given an outline, is essentially the same as that employed by PTOLEMY (second century A.D.).* Ptolemy, however, also made use of the inscribed pentagon (cf. Exercise XLIX, Ex. 12), and his table was a table of chords, not of sines. (See Art. 70.) His table gives the values of the chord for each half degree of arc with a degree of accuracy somewhat greater than that which would correspond to a modern five-place table. The earlier tables of HIPPARCHUS and MENELAUS are not extant.

The Hindus followed the method which we have outlined even more closely. In fact, the table given by Āryabhata (born 476 A.D.) gives the values of the sine at intervals of 3° 45′. As we have seen, this is precisely the interval which would arise as a result of continued bisection of 30°.

Essentially the same method was used in subsequent improvements and enlargements of these tables, especially by RHETICUS (1514–1574) and PITISCUS (1561–1613). Other far more powerful methods have since been developed, based essentially on the notions of the calculus and the theory of infinite series.

EXERCISE XLIX

1. From the functions of 30° find those of 60°.

2. From the functions of 60° find those of 120°.

3. From the functions of 90° find those of 45°.

4. From the functions of 30° find those of 15°.

5. From the functions of 15° find those of 7°.5.

6. Find formulæ for $\sin 3\alpha$, $\cos 3\alpha$, $\tan 3\alpha$.

HINT. Put $3\alpha = 2\alpha + \alpha$.

7. Find formulæ for $\sin 4\alpha$, $\cos 4\alpha$, $\tan 4\alpha$.

8. Find formulæ for $\sin 5\alpha$, $\cos 5\alpha$, $\tan 5\alpha$.

9. Prove formula (1), Art. 83, by means of a figure.

10. Given $\tan\theta = \frac{4}{3}$, θ being in the first quadrant. Find the functions of 2θ and $\frac{1}{2}\theta$.

---

* Ptolemy's great work on Astronomy, usually known as the *Almagest*, remained in undisputed authority until the time of *Copernicus*. The so-called Ptolemaic system of astronomy, as opposed to the more modern Copernican system, was named after him for this reason.

11. If θ is in the third quadrant and $\sin\theta = \dfrac{-2}{\sqrt{5}}$, find the functions of 2θ.

12. In a circle of radius 1, inscribe a regular pentagon. Show that, by means of this construction the trigonometric functions of 72° and 18° may be computed. In particular, show that

$$\sin 18^\circ = \tfrac{1}{4}(\sqrt{5} - 1).$$

13. Making use of the results of Ex. 12, compute the functions of 12°.

14. Making use of the results of Ex. 13, compute the functions of 6°.

Assuming the truth of the law of cosines, and setting $s = \frac{1}{2}(a + b + c)$, prove the following formulæ for the functions of the half angles of a triangle.

15. $\sin \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{bc}}.$

16. $\cos \frac{1}{2} A = \sqrt{\dfrac{s(s-a)}{bc}}.$

17. $\tan \frac{1}{2} A = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}.$

18. Prove formulæ (6) and (7) of Art. 84 by means of equations (4).

85 a. The limit $\dfrac{\sin\theta}{\theta}$ and related limits. Although the method sketched in Art. 84 for calculating a table of the values of the trigonometric functions is adequate, it involves far more labor than is actually necessary. The following theorem, whose truth is almost self-evident, is of great importance in this connection as it enables us to calculate the sine of a very small angle with a minimum of effort.

If an angle or arc is expressed in radians, the quotient $\dfrac{\sin\theta}{\theta}$ *approaches the limit* 1 *when the angle itself approaches zero as a limit. In symbols*

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1.$$

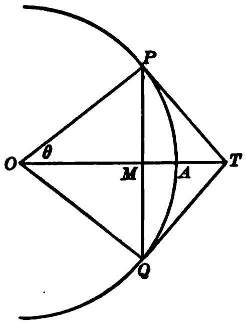

FIG. 104

In order to prove this theorem, let us draw an acute angle $AOP = \theta$ as in Fig. 104, and symmetrically the angle AOQ also equal to θ. With the vertex O as center and any convenient radius r, draw the circular arc PAQ and join PQ, intersecting OA in M. The tangents PT and QT, at P and Q, will meet in a point T of OA prolonged.

Obviously we shall have

(1) $PQ < \text{arc } PAQ < PT + TQ.$

But

$$PQ = 2\,PM = 2\,r \sin \theta,$$

$$\text{arc } PAQ = 2 \text{ arc } AP = 2\,r\theta \quad \text{(Art. 74, equation (5))},$$

$$PT + TQ = 2\,PT = 2\,r \tan \theta,$$

where the truth of the second equation depends essentially upon our assumption that θ is expressed in radians. If these values be substituted in (1), we find

$$2\,r \sin \theta < 2\,r\,\theta < 2\,r \tan \theta,$$

or, after division by the positive quantity $2\,r$,

(2) $$\sin \theta < \theta < \tan \theta.$$

If we divide all three members of this inequality by the positive number $\sin \theta$, we find

(3) $$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}.$$

We know that $\cos \theta$ approaches the limit 1 when θ approaches zero. Since the value of $\frac{\theta}{\sin \theta}$, according to (3), lies between 1 and $\frac{1}{\cos \theta}$, which latter quantity itself approaches 1, $\frac{\theta}{\sin \theta}$ must also have 1 as its limit. That is,

(4) $$\lim_{\theta \doteq 0} \frac{\theta}{\sin \theta} = 1.$$

From (3) we have further

$$1 > \frac{\sin\theta}{\theta} > \cos\theta,$$

so that, by a similar argument, we find

$$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta} = 1. \tag{5}$$

If we divide all members of (2) by the positive quantity $\tan\theta$, we find

$$\cos\theta < \frac{\theta}{\tan\theta} < 1,$$

so that

$$\lim_{\theta \doteq 0} \frac{\theta}{\tan\theta} = \lim_{\theta \doteq 0} \frac{\tan\theta}{\theta} = 1. \tag{6}$$

Since

$$\frac{\sin(-\theta)}{-\theta} = \frac{-\sin\theta}{-\theta} = \frac{\sin\theta}{\theta},$$

$$\frac{\tan(-\theta)}{-\theta} = \frac{-\tan\theta}{-\theta} = \frac{\tan\theta}{\theta},$$

equations (4), (5), and (6) will still remain true if θ approaches zero through negative instead of positive values.

These formulæ have many important applications. For the present, we shall mention only the one indicated at the beginning of this article. The content of equation (5) may be formulated as follows. If we write

$$\frac{\sin\theta}{\theta} = 1 - \delta, \tag{7}$$

the angle θ (expressed in radians) may be chosen so small that δ $\left(\text{the difference between 1 and } \frac{\sin\theta}{\theta}\right)$ will become less than any previously assigned quantity. Since we find from (7)

$$\sin\theta - \theta = -\delta\theta,$$

we see that we can make the angle θ (expressed in radians) so small that the difference between θ and $\sin\theta$ becomes less than any previously assigned small fraction of θ.

Suppose, for instance, that we wish to compute the sine of a small angle to 5 decimal places, that is, with an error which shall be less than 5 units of the sixth decimal place, or .000005. We now know that the angle may be chosen so small that its sine may be equated to the radian measure of the angle itself with an error of less than 5 units of the sixth decimal place. In other words, the equation

$$(8) \qquad \sin \theta = \theta \text{ (in radians)}$$

will be true up to five decimal places for all angles which are sufficiently small.

Of course, our method does not inform us just *how* small θ must be in order that equation (8) may be true up to five decimal places. It would take us too far afield to investigate this question, a complete answer to which is beyond the scope of this book. The student may convince himself, however, by actual comparison with the tables, that equation (8) is true to five decimal places for all angles less than 2°. In all numerical work, then, involving such small angles, no error noticeable in five-place calculations is introduced by putting $\sin \theta = \theta$ (in radians).

Since we have

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \text{ (Art. 84, equation (1))},$$

we may put

$$(9) \qquad \cos \theta = 1 - 2\left(\frac{\theta}{2}\right)^2 = 1 - \frac{1}{2}\theta^2,$$

a formula which will certainly be true up to the fifth decimal place for all angles less than 2°. In fact equation (9) holds to five decimal places even for angles much larger than this and may serve for the purpose of computing the cosines of such angles. Since θ^2 is small as compared with θ, if θ itself is small, we shall even be justified in equating $\cos \theta$ to 1 for very small angles. Our tables show that no error is introduced in five-place calculations by putting $\cos \theta = 1$, if θ is less than 0° 16′.

Since (8) is true to five decimal places if $\theta < 2°$, we see that we shall have

$$\sin 2\theta = 2\theta$$

with the same degree of approximation if $\theta < 1°$. More generally the formula

$$\sin n\theta = n\theta \ (\theta \text{ in radians}) \tag{10}$$

is correct to five decimal places if $n\theta$ is less than 2°.

The results deduced in this article make it very easy to compute the functions of very small angles. By combining these results with the methods of Art. 83 an extensive table of the trigonometric functions may be constructed with comparative ease.

EXERCISE L

Compute the values of the following functions of small angles to five decimal places by the method of Art. 85 *a* and compare with the values obtained from the table:

1. $\sin 12'$. **3.** $\sin 1°$. **5.** $\tan 1°$.

2. $\tan 15'$. **4.** $\cos 1°$. **6.** $\cot 1°$.

7. What will be the angle subtended by a lamp-post 10 feet high at a distance of one mile?

8. In order to find the distance from the earth to the moon, the following plan may be adopted. Two astronomers stationed at A and B respectively (Fig. 105) observe at the same instant the angular distance of the moon's center M from their respective zeniths (their overhead points), Z and Z'. This gives the angles

$$\alpha = ZAM \text{ and } \beta = Z'BM.$$

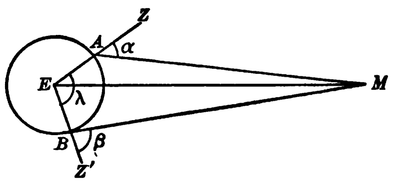

FIG. 105

For the sake of simplicity assume that both stations A and B are on the equator, that the moon is in the plane of the equator, and let E be the center of the earth. Then $\angle AEB = \lambda$ is equal to the difference between the longitudes of the two stations and may be regarded as known. We may now compute

$$\angle EAM = 180° - \alpha, \ \angle EBM = 180° - \beta, \ \angle AEB = \lambda.$$

Now the sum of the four angles of the quadrilateral $AEBM$ is four right angles; that is,

$$\angle M + 360° - \alpha - \beta + \lambda = 360°,$$

whence

$$\angle M = \alpha + \beta - \lambda.$$

From the value of M obtained in this way, it is easy to compute the angle subtended by the earth's radius at the center of the moon. This angle is called the *moon's parallax.*

Find the moon's distance from the earth if the moon's parallax is 57′ and if the earth's radius is 4000 miles.

9. The apparent diameter of the moon as seen from the earth is about 31′. Making use of the result of Ex. 8, what is the moon's diameter in miles?

10. The sun's parallax is about 8″.8. Assuming 4000 miles as the length of the earth's radius, find the distance from the earth to the sun.

85 *b*. The auxiliary quantities S and T. We have seen in Art. 28 that the ordinary tables of sines and tangents become inconvenient for very small angles. To avoid this inconvenience, we constructed an additional table (Table III), giving the values of the sines and tangents of such small angles directly for every second of arc. But we may accomplish the same purpose in another way, by means of an auxiliary table occupying far less space than the additional table just mentioned. This second method is based on the fact that the quotients

$$\frac{\sin\theta}{\theta} \text{ and } \frac{\tan\theta}{\theta}$$

change very slowly if θ is a small angle.

We have just seen that each of these quotients has unity as its limit when θ approaches zero, provided that the angle is measured in radians. Let us instead express θ in minutes of arc. Let θ' denote the number of minutes and $\theta^{(R)}$ the number of radians contained in the angle θ. Then, according to Art. 85 *a*,

$$(1) \qquad \underset{\theta \doteq 0}{\text{limit}} \frac{\sin\theta}{\theta^{(R)}} = \underset{\theta \doteq 0}{\text{limit}} \frac{\tan\theta}{\theta^{(R)}} = 1.$$

Since (Art. 74)

$$1^\circ = \frac{\pi}{180} \text{ radians} = 0.0174533 \text{ radian},$$

and therefore

$$1' = 0.0002909 \text{ radian},$$

the angle θ, which contains θ' minutes, will contain

$$\theta^{(R)} = 0.0002909\,\theta' \text{ radian}.$$

Consequently we find

$$\frac{\sin\theta}{\theta'} = \sin\theta \div \frac{\theta^{(R)}}{0.0002909} = 0.0002909\,\frac{\sin\theta}{\theta^{(R)}},$$

and therefore, on account of (1),

(2) $$\lim_{\theta \doteq 0} \frac{\sin\theta}{\theta'} = \lim_{\theta \doteq 0} \frac{\tan\theta}{\theta'} = 0.0002909.$$

In other words, if θ' is an angle expressed in minutes, the common limit toward which $\frac{\sin\theta}{\theta'}$ and $\frac{\tan\theta}{\theta'}$ tend, when θ' approaches zero, is a number whose first seven decimal places are given by 0.0002909.

Let us write

(3) $$s = \frac{\sin\theta}{\theta'}, \qquad t = \frac{\tan\theta}{\theta'}.$$

These quantities change their values very slowly for small values of θ. In fact we have just seen that, for angles which are small enough, we shall have

$$\log s = \log t = \log 0.0002909 = 6.46373 - 10.$$

For $\theta = 2^\circ = 120'$ we have

$$s = \frac{\sin 2^\circ}{120}, \qquad t = \frac{\tan 2^\circ}{120},$$

which gives, if we look up the logarithms from the tables,

$$\begin{array}{rl} \log\sin 2^\circ = & 8.54282 - 10 \\ \log 120 = & 2.07918 \\ \hline \log s = & 6.46364 - 10 \end{array} \qquad \begin{array}{rl} \log\tan 2^\circ = & 8.54308 - 10 \\ \log 120 = & 2.07918 \\ \hline \log t = & 6.46390 - 10 \end{array}$$

Therefore, while θ changes from 0° to 2°, $\log s$ changes only by 9 units and $\log t$ by 17 units of the fifth decimal place.

Table IV enables us to find the values of $S = \log s$ and $T = \log t$ for every angle between 0° and 2°. To find the logarithm of the sine or tangent of such an angle we have the formulæ (immediate consequences of (3))

$$(4)\qquad \begin{aligned} \log\ \sin\theta &= \log\theta' + \log s = \log\theta' + S, \\ \log\ \tan\theta &= \log\theta' + \log t = \log\theta' + T. \end{aligned}$$

If the log sin or log tan of a small angle is given, to find the angle, we may do this by means of one of the equations

$$(5)\qquad \begin{aligned} \log\theta' &= \log\sin\theta - S, \\ \log\theta' &= \log\tan\theta - T, \end{aligned}$$

obtained from (4) by transposition.

Of course the quantities S and T are available, not only for sines and tangents of small angles, but also for cosines and cotangents of angles close to 90°.

EXERCISE LI

1. Find the sine and tangent of 1° 13′.21 by using the auxiliaries S and T.

Solution. Since $\theta = 1°\ 13'.21$, we have $\theta' = 73'.21$.

$$\begin{array}{rl} \log\theta' &= 1.86457 \\ S &= 6.46369 - 10 \\ \hline \log\sin\theta &= 8.32826 - 10 \end{array} \qquad \begin{array}{rl} \log\theta' &= 1.86457 \\ T &= 6.46379 - 10 \\ \hline \log\tan\theta &= 8.32836 - 10 \end{array}$$

2. Given $\log\sin\theta = 8.24798 - 10$. Find θ.

Solution. We find from Table IV corresponding to $\log\sin\theta = 8.24798 - 10$, $S = 6.46370$. Formula (5) leads to the calculation

$$\begin{array}{rl} \log\sin\theta &= 8.24798 - 10 \\ S &= 6.46370 - 10 \\ \hline \log\theta' &= 1.78428 \end{array} \qquad \therefore\ \theta' = 60'.85 = 1°\ 0'.85.$$

Find the values of the logarithms of the following functions by means of the auxiliaries S and T:

3. sin 1° 21′.63.

4. tan 0° 32′.61.

5. cos 89° 13′.21.

6. cot 88° 21′.75.

Find the angles determined by the following functions by means of S and T:

7. $\log\sin\theta = 7.76345 - 10$.

8. $\log\tan\theta = 8.50731 - 10$.

9. $\log\cos\theta = 8.42371 - 10$.

10. $\log\cot\theta = 8.53729 - 10$.

CHAPTER XII

DIRECTED LINES AND DIRECTED LINE-SEGMENTS *

86. Plan of another proof for the addition formulæ. When we proved the addition theorem in Art. 79, we found it necessary to divide the proof into a number of cases according as the angles were in the first, second, third, or fourth quadrants. To be sure, by making use of the method of mathematical induction we found it a fairly simple matter to make an exhaustive discussion covering all cases. Nevertheless we feel that it must be possible to devise a method enabling us to prove this theorem at one stroke for angles of any magnitude. The key to the solution of this problem is found to be a careful formulation of the notions of a *directed line* and a *directed line-segment*. Since these notions are of very great importance, not only in this connection, but in many other parts of pure and applied mathematics, we shall find it worth our while to speak of them, even if they are not absolutely indispensable for the proof of the addition theorem.

87. Directed lines and segments. A straight line is infinite in extent and is determined by any two distinct points upon it. We may, however, think of one and the same straight line as having either of two opposite directions, in which case we speak of it as **a directed line**. Since we can never draw more than a finite portion of a line, we may indicate the direction of a directed line by placing a + sign near one end of that portion which actually appears in the figure. In Fig. 106 we have thus indicated the direction of

* This chapter may be omitted in a first course if the time is insufficient.

the directed line l which is to be thought of as pointing toward the upper right-hand corner of the page.

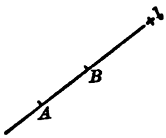

FIG. 106

This method of indicating the direction of a directed line has already been used in this book to indicate the positive direction of the x-axis and y-axis of a system of rectangular coördinates. (See Art. 63.) These are directed lines.

A **line-segment** is a finite portion of a line and may be described by naming its end points, such as AB in Fig. 106. But again, we may think of it as a **directed line-segment**, thus distinguishing between AB and BA.

When a directed line-segment lies upon a directed line, its direction or **sense** may be the same as that of the line or else opposite to it. If a directed line-segment on l is 5 units long and if its direction is the same as that of l, we may represent it by the number $+5$. A line-segment of the same length and opposite direction will be represented by -5. In general, *a directed line-segment, which lies on a directed line, shall be counted positive or negative according as it has the same or the opposite direction as the directed line.* For such line-segments, we always have $BA = -AB$, or $AB + BA = 0$.

We are now ready to prove the following theorem. *If A, B, C, are any three points on a directed line, then*

(1) $$AB + BC = AC,$$

where AB, BC, and AC are directed line-segments.

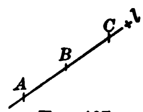

FIG. 107

PROOF. 1. Let AC be positive and let B be between A and C. Then AB and BC are also positive and the truth of the theorem, in this case, is obvious. (See Fig. 107.)

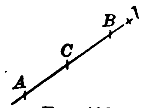

FIG. 108

2. Let AC be positive, but let C be between A and B. Then AB is positive, but BC is negative and equal to $-CB$. Thus

$AB + BC = AB - CB = AC$. (See Fig. 108.)

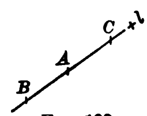

FIG. 109

3. Let AC still be positive, but let A be between B and C. Then (Fig. 109),

$$AB + BC = -BA + BC = BC - BA = AC.$$

If AC is negative, there are again three cases according as the order of the three points is CBA, BCA, or CAB. But the above relation (1) will be found to be true in all cases. These last three cases may, of course, also be reduced to the former three by reversing the direction of the line l.

The following is a simple corollary of the above theorem. *If A, B, C, D, are any four points of a directed line, we have the relation*

$$(2) \qquad AB + BC + CD = AD$$

between the directed line-segments AB, BC, CD, and AD.

In fact, by the theorem just proved, we have

$$AB + BC = AC, \qquad AC + CD = AD,$$

so that we find by addition

$$AB + BC + AC + CD = AC + AD,$$

which reduces to (2) if we subtract AC from both members.

It may now be proved by induction that, in general, *if A, B, C, $\cdots$ M, N are any finite number of points on a directed line, then*

$$(3) \qquad AB + BC + CD + \cdots + MN = AN.$$

Equations (1), (2), (3) may also be proved by algebra. On the directed line l, let us introduce a point O as origin or zero point of a scale, whose positive readings are on that side of O which corresponds to the positive direction of the line l. This is precisely what we did when we established scales upon the x-axis and y-axis of a coördinate system (Art. 63). Denote by l_A the reading of the scale which corresponds to the point A, by l_B that which corresponds to B. The difference $l_B - l_A$ will give the length of the line-segment AB, affected with a plus or minus sign according as AB is a positive or negative line-segment in the sense of our previous definition.

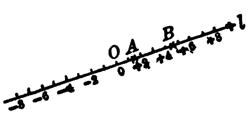

FIG. 110

If then we have any three points A, B, C on the directed line, we shall have

$$AB = l_B - l_A, \qquad BC = l_C - l_B, \qquad AC = l_C - l_A,$$

and therefore

$$AB + BC = l_B - l_A + l_C - l_B = l_C - l_A = AC,$$

which is the same as (1). In the same way we may also prove equations (2) and (3).

88. Angles between directed lines. Let l and m be two directed lines, and let us denote the angle between their positive directions by (l, m) or (m, l) according as we think of l or m as the initial side of the angle. To be perfectly specific, we understand by (l, m) the angle, less than 360°, through which it would be necessary to rotate the directed line l in the counter-clockwise direction, in order to make its positive direction coincide with the positive direction of the directed line m. In Fig. 111, the angle (l, m) is marked θ. Similarly (m, l) is the angle through which m would have to be turned in the positive (counter-clockwise) direction in order to make the positive direction of m coincide with that of l. In Fig. 111 (m, l) is marked θ'.

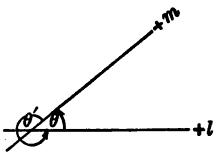

FIG. 111

We see that we shall always have

(1) $$(l, m) + (m, l) = 360°,$$

so that

$$(m, l) = 360° - (l, m)$$

and therefore (see Art. 77, equations (7)),

(2) $$\sin (m, l) = -\sin (l, m), \qquad \cos (m, l) = \cos (l, m).$$

89. Projections. The projection of a point P on a line l is the foot of the perpendicular dropped from the point to the line. The projection of a line-segment AB on a line l (see Fig. 112) is the line-segment $A'B'$ of l bounded by the projections of A and B. If AB is a *directed* line-segment, so is $A'B'$.

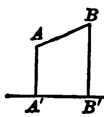

FIG. 112

We wish to solve the following problem. *Given a directed line-segment AB on a directed line p; to find the magnitude and sign of its projection upon any second directed line l.*

The line-segment AB may have the same sense as the directed line p or else the opposite sense; it will be represented by a positive number in the first case and by a negative number in the second (Art. 87). If we denote by $|AB|$ the *positive* number which represents merely the length (regardless of direction) of the line-segment AB, we shall have

(1) $\qquad AB = |AB|$, read $AB =$ *length* AB,

or

(2) $\qquad AB = -|AB|$, read $AB =$ *minus length* AB,

according as the direction of AB agrees with that of the directed line p or not.

Let us consider first the case (1) in which AB is positive. (See Figs. 113 and 114.) Let OM be the projection of AB on l. Choose O as origin and l as the x-axis of a system of

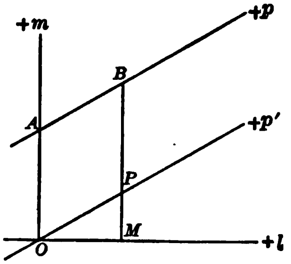

FIG. 113

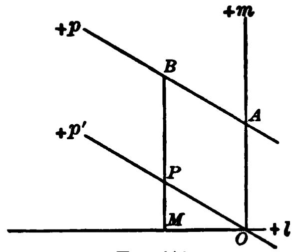

FIG. 114

coördinates, so that the positive x-axis coincides with the positive direction of the line l. The line m through O, perpendicular to l and with its positive direction as indicated in the figures, will then be the y-axis. Through O draw a directed line p' parallel to p and let P be its intersection with the line MB through B perpendicular to l.* The line-

* The word *parallel* in this theory means, not only that the lines are parallel in the ordinary sense, but that their positive directions are the same. The word *anti-parallel* is sometimes used for two parallel directed lines whose positive directions are opposite.

segments AB and OP have the same length and direction, and the same directed line-segment OM on l as projection. But, by the definition of the cosine of a general angle (Art. 64),

$$\cos(l, p) = \cos(l, p') = \frac{OM}{OP},$$

since OM is the abscissa and OP the radius vector of a point P on the terminal side of the angle (l, p'), this angle being in its standard position and $OP = AB$ being positive in the case under consideration. Consequently we have

$$OM = OP \cos(l, p) = AB \cos(l, p).$$

Since OM is the projection of AB on l, we may write this as follows:

(3) $\quad OM = \text{proj}_l\, AB = AB \cos(l, p) = AB \cos(p, l);$*

for, according to equation (2), Art. 88, the angles (l, p) and (p, l) have the same cosine.

Thus the projection upon the directed line l, of the positive line-segment AB of the directed line p, is equal to AB multiplied by the cosine of the angle between l and p. The projection is a directed line-segment on l, positive if (l, p) is in the first or fourth, negative if (l, p) is in the second or third quadrant.

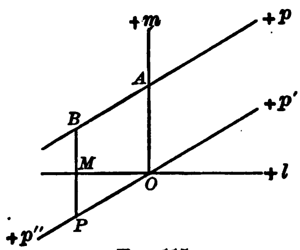

FIG. 115

We have proved equation (3) when AB is positive. Let us consider now the case when AB is a negative line-segment of p. (See Fig. 115.) The projection of AB, as well as that of OP, is OM. We may think of OP, which is a negative line-segment on p', as a positive line-segment of a directed line p'', coincident with p' but having the opposite direction. But this latter directed line makes an angle with l just 180° greater than (l, p'); that is,

(4) $$(l, p'') = (l, p') + 180°.$$

* The symbol $\text{proj}_l\, AB$ is read "projection of AB upon l."

For if we rotate l in the counterclockwise direction around O as center, it will take 180° more to make $+l$ coincide with $+p''$ than with $+p'$. According to formula (3), which we know to be valid for all positive line-segments, we have therefore

$$\text{proj}_l\, AB = |AB| \cos(l, p''),$$

where $|AB|$ denotes the *length* of the line-segment AB taken as a positive number. But according to (4) and equations (4) of Art. 77, we have

$$\cos(l, p'') = -\cos(l, p') = -\cos(l, p) = -\cos(p, l),$$

so that

$$\text{proj}_l\, AB = -|AB| \cos(l, p) = -|AB| \cos(p, l).$$

Since in our case AB was a negative line-segment, we had (cf. equation (2))

$$AB = -|AB|,$$

so that we may write finally

$$(3) \qquad \text{proj}_l\, AB = AB \cos(l, p) = AB \cos(p, l).$$

In other words, *formula* (3) *for the projection upon a directed line l, of a directed line-segment AB of a second directed line p, is true for both positive and negative line-segments.*

If we think of the line-segments as mere lengths, not endowed with direction, formula (3) may be simplified to

$$A'B' = \text{proj}_l\, AB = AB \cos\theta,$$

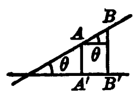

FIG. 116

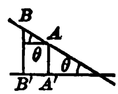

FIG. 117

where θ is the angle between AB and the line l, this angle being understood in the sense of elementary geometry without any reference to a positive sense of rotation. Consequently θ may be regarded as an acute angle, so that its cosine will always be positive. (See Figs. 116 and 117.)

90. Projection of a broken line. Let us connect two points A and C by a directed line-segment AC. We may think of

this line-segment as describing the shortest path from A to C in magnitude and direction. Let ABC be a second (longer) path from A to C made up of two directed line-segments AB and BC (Fig. 118).

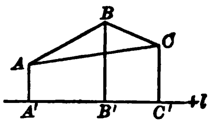

FIG. 118

Let us project these three line-segments on any directed line l, so that

(1) $A'B' = \text{proj}_l\, AB,\ B'C' = \text{proj}_l\, BC,\ A'C' = \text{proj}_l\, AC.$

Since $A'B'$, $B'C'$ and $A'C'$ are three directed line-segments of a directed line, we have (Art. 87, equation (1))

$$A'C' = A'B' + B'C';$$

whence, substituting the values (1),

(2) $$\text{proj}_l\, AC = \text{proj}_l\, AB + \text{proj}_l\, BC.$$

By means of the more general relation (3) of Art. 87, we may prove the following general theorem, of which (2) expresses the simplest special case.

Let us connect two points by two paths, one of which is a single directed line-segment, while the other is made up of a finite number of directed line-segments. Then the projection, upon any directed line, of the first path is equal to the sum of the projections of the various line-segments which make up the second path.

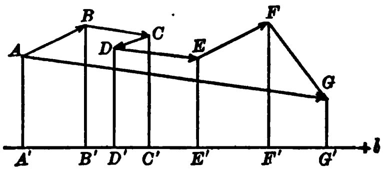

FIG. 119

Figure 119, in which the various line-segments are marked with arrowheads to indicate their directions, illustrates this theorem. Since the positive direction of l is toward the right, the projections $A'G'$, $A'B'$, $B'C'$, $D'E'$, $E'F'$, $F'G'$ are all positive, while the projection $C'D'$ of CD is negative. Consequently

$$\text{proj}_l\, AB + \text{proj}_l\, BC + \text{proj}_l\, CD$$
$$= A'B' + B'C' + C'D' = A'B' + B'C' - D'C' = A'D'.$$

Let the line l in Fig. 118 coincide with AC, and denote by a, b, c the lengths of the three line-segments BC, AC, and

AB respectively. (See Fig. 120.) According to (2) and Art. 89, equation (3), we shall have

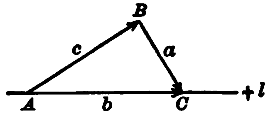

FIG. 120

(3) $b = c \cos A + a \cos C,$

a relation which may be used to advantage in many problems concerning triangles. Of course the two equations similar to (3)

$$(4) \qquad \begin{aligned} c &= a \cos B + b \cos A, \\ a &= b \cos C + c \cos B, \end{aligned}$$

may be proved in the same fashion.

91. Direction cosines of a line. Let l' be a directed line through the origin of coördinates. The angles (x, l') and (l', y), which its positive direction makes with the positive x-axis and y-axis, are called its *direction angles.* The direction angles (x, l) and (l, y) of a line l which does not pass through the origin are defined to be the same as those of a parallel line l' which does pass through the origin. Observe that the angle (x, l) has the x-axis as initial side, while the initial side of the second direction angle (l, y) is not the y-axis, but the line l.

If the angle (x, l) is in the first quadrant (Fig. 121), we clearly have

$$(1) \qquad (x, l) + (l, y) = 90°.$$

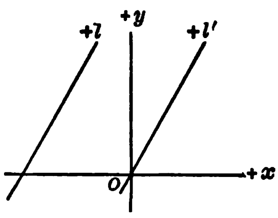

FIG. 121

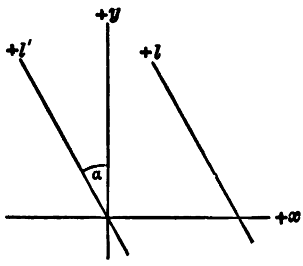

FIG. 122

If the angle (x, l) is in the second quadrant (Fig. 122), we have

$$\begin{aligned} (x, l) &= (x, l') = 90° + \alpha, \\ (l, y) &= (l', y) = 360° - \alpha. \end{aligned}$$

For clearly it requires a positive (counterclockwise) rotation of $360° - \alpha$ to make $+l'$ coincide with $+y$. Therefore we have, in this case,

(2) $$(x, l) + (l, y) = 360° + 90°,$$

and the same relation holds if (x, l) is in the third or fourth quadrant, as may be verified easily. Thus we always have either $(l, y) = 90° - (x, l)$ or $(l, y) = 360° + 90° - (x, l)$, so that in all cases

(3) $$\cos(y, l) = \cos(l, y) = \sin(x, l), \quad \sin(l, y) = \cos(x, l).$$

The two quantities $\cos(x, l)$ and $\cos(y, l)$ are called the *direction cosines of the directed line l.*

Since we have, for any angle,

$$\cos^2(x, l) + \sin^2(x, l) = 1,$$

we find from (3) *the following simple relation between the direction cosines of a line l;*

(2) $$\boldsymbol{\cos^2(x, l) + \cos^2(y, l) = 1.}$$

92. Formula for the cosine of the angle between two lines whose direction cosines are given. Let us consider two directed lines l and m whose direction cosines are $\cos(x, l)$, $\cos(y, l)$, and $\cos(x, m)$, $\cos(y, m)$ respectively. We wish to find a formula for the cosine of the angle between the two lines.

We may assume that the lines l and m pass through the origin. If they do not, we may first solve the problem for two lines l', m', parallel to l, m respectively, which do pass through the origin. Since l', m' have the same direction cosines as l and m, and since the angle between l', m' is equal to that between l, m, the two problems are clearly equivalent.

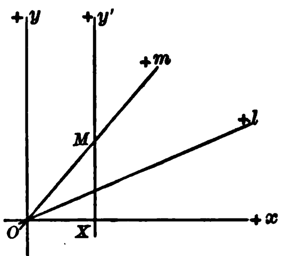

FIG. 123

Let us choose a point M (see Fig. 123), on the line m, such that the line-segment OM is positive. Let X be the projection of M on the x-axis. Then the projection of the broken

line OXM on l will be equal to that of OM (Art. 90, equation (2)). That is,

$$(1) \qquad \text{proj}_l\, OM = \text{proj}_l\, OX + \text{proj}_l\, XM.$$

On account of Art. 89, equation (3), we have

$$(2) \qquad \begin{aligned} &\text{proj}_l\, OM = OM \cos(m, l), \\ &\text{proj}_l\, OX = OX \cos(x, l), \quad \text{proj}_l\, XM = XM \cos(y, l), \end{aligned}$$

since XM is a directed line-segment on a directed line y' parallel to the y-axis.*

Substitution of (2) in (1) gives

$$(3) \qquad OM \cos(m, l) = OX \cos(x, l) + XM \cos(y, l).$$

But
$$\begin{aligned} OX &= \text{proj}_x OM = OM \cos(x, m), \\ XM &= \text{proj}_{y'} OM = \text{proj}_y OM = OM \cos(y, m), \end{aligned}$$

since the projections of OM, on the two parallel directed lines y and y', are equal. If we substitute these values in (3), we find

$$OM \cos(m, l) = OM \cos(x, l) \cos(x, m) + OM \cos(y, l) \cos(y, m)$$

or, upon division by OM,

$$(4) \qquad \boldsymbol{\cos(m, l) = \cos(x, l) \cos(x, m) + \cos(y, l) \cos(y, m)},$$

the desired formula.

The proof which we have given of this formula is perfectly general; that is, it is applicable, no matter how large or small the angles (l, m), (x, l), etc., may be, in what quadrants they happen to lie, or whether they are positive or negative. In fact, the figure (Fig. 123) has not really been used in the demonstration except for the purpose of suggesting the order of the various steps of the argument. Every step of this proof can be justified by quoting a previously demonstrated general theorem.

It will be a good exercise for the student to repeat the argument with a different figure in which some or all of the angles concerned are not acute.

---

* In Fig. 123 we have chosen the positive direction of y' to correspond to that of y. This is not essential, but it is convenient.

93. New proof for the addition and subtraction formulæ. Let us denote the angles (x, l) and (x, m) by α and β respectively. Then

$$\cos(l, m) = \cos(\alpha - \beta),$$

and (Art. 91, equations (3)),

$$\cos(y, l) = \sin(x, l) = \sin\alpha, \quad \cos(y, m) = \sin(x, m) = \sin\beta.$$

Consequently we find, from equation (4) of Art. 92,

$$(1) \qquad \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$

the subtraction formula for the cosine. We shall leave it as an exercise for the student to deduce from this the subtraction formula for the sine and the addition formulæ for both functions. (See Exercise LII.)

94. The generalized law of sines. If we attribute a definite direction to every line of the plane, and define angles between them, as in Art. 88, with reference to a positive direction of rotation, the angle between any two such directed lines may be greater than 180° and the trigonometric functions of such angles will have definite signs as well as numerical magnitude. Moreover, every line-segment will then also have a definite sign.

It may be shown that the law of sines, when written in the form

$$(1) \qquad \frac{BC}{\sin(b, c)} = \frac{CA}{\sin(c, a)} = \frac{AB}{\sin(a, b)}$$

will be true of any triangle, not merely with regard to the magnitudes of the quantities involved, but also with regard to their signs.

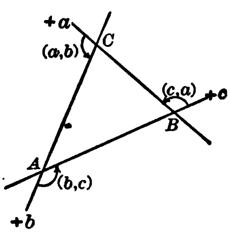

FIG. 124

In order to explain this statement, consider first the case that the positive directions of the three directed lines a, b, c (the sides of the triangle) are from B toward C, from C toward A, and from A toward B respectively. (Cf. Fig. 124.) Then BC, CA, and AB are all positive,

and the three denominators in (1) are either all positive or all negative.

In Fig. 124 they are all positive. But they would all be negative if the clockwise rotation had been chosen as positive direction of rotation. They would all be negative, even without any change in the choice of the positive direction of rotation, if the points named A and B in Fig. 124 were interchanged.

Consequently equations (1) are true in this case, not merely numerically, but also with regard to sign.

Let us now invert the positive direction of a single one of the lines, that of a, for instance. Then BC becomes negative; the angle (b, c) and the segments CA and AB remain unaltered; (c, a) and (a, b) each change by 180°, so that their sines change sign. Consequently equations (1) will still be verified.

By combining several such changes we easily arrive at the conclusion that equations (1) will always be true, in magnitude and sign, no matter how the positive directions of the three lines a, b, c may have been selected or which of the two opposite kinds of rotation be regarded as positive.

This generalization of the law of sines is due to the great geometer Möbius (1790–1868), and is of great importance in many applications, especially in projective geometry.

EXERCISE LII

1. Show that the principal theorem of Art. 90 may be enunciated as follows. If a finite number of directed line-segments form a closed polygon, the sum of their projections upon any directed line is equal to zero.

2. Show that the x-component of the resultant of two forces (cf. Art. 58) is equal to the sum of the x-components of the two original forces. Similarly for the y-components.

3. From formula (1) of Art. 93 deduce the formula for $\sin(\alpha - \beta)$.

4. From formula (1) of Art. 93 deduce the formulæ for $\cos(\alpha + \beta)$ and $\sin(\alpha + \beta)$.

5. Generalize the law of tangents from the standpoint of directed line-segments in a way which shall be analogous to the generalization of the law of sines carried out in Art. 94.

Hint. The law of tangents may be deduced from the law of sines.

6. Generalize in similar fashion the projection formulæ (3) and (4) of Art. 90.

CHAPTER XIII

THE INVERSE TRIGONOMETRIC FUNCTIONS AND TRIGONOMETRIC EQUATIONS

95. The problem of inverting the trigonometric functions. In the light of our recent studies we may say, with a considerable degree of propriety, that *trigonometry is a discussion of the properties of the trigonometric functions.* However, our discussion of these functions, so far, has been somewhat one-sided. With the exception of a few practical applications in the first part of the book, we have always looked upon these functions from the point of view of what may be called the *direct problem;* that is, given the angle, to find the function. We now propose to discuss, somewhat more fully than has been done so far, the *inverse problem;* that is, given the value of one of the functions, to find the corresponding angles.

We notice at once a fundamental difference between these two problems, which may be illustrated by the relation between an angle and its sine. Every angle has only one sine, but there are many angles which have the same sine. Consequently while *the direct problem* (to find the sine of a given angle) *has only one solution, the inverse problem* (to find an angle with a given sine) *has many solutions.*

96. Determination of all of the angles which correspond to a given value of one of the functions. When we are solving triangles, the angles are necessarily either in the first or second quadrant, so that in most cases we experience little difficulty in finding a unique angle as an answer to a given problem of this kind. But, even in this restricted field, we found that a problem may have *two* solutions, owing to the fact that

there exists two angles θ, one acute and one obtuse, corresponding to a given positive value of $\sin\theta$. (Cf. Art. 55.)

If the sine of an angle is given, and no particular quadrant or set of quadrants is prescribed for the angle, the number of values which the angle may have is unlimited.

We may prove this and find out how all of these angles are related to each other as follows:

Let the given value of the sine be denoted by s. If s is numerically greater than unity, the problem has no solution, since no angle has a sine numerically greater than 1. If s is numerically less than 1 and positive, there exists an acute angle θ and an obtuse angle $180° - \theta$, such that

$$\sin\theta = \sin(180° - \theta) = s.$$

But, on account of the periodic character of the sine, we also have

$$\sin(\theta + n \cdot 360°) = \sin\theta = s,$$
$$\sin(180° - \theta + n \cdot 360°) = \sin(180° - \theta) = s,$$

where n is any positive or negative integer or zero. We see therefore that all angles, such as

$$(1) \qquad n \cdot 360° + \theta = 2n \cdot 180° + \theta,$$

or

$$(2) \qquad n \cdot 360° + 180° - \theta = (2n + 1) \cdot 180° - \theta,$$

have the same sine as the angle θ. This may be expressed as follows. All of those angles which can be obtained by adding a given angle to any even multiple of 180° or else by subtracting the given angle from any odd multiple of 180°, have the same sine.

That these are the only angles which have the same sine follows easily from the fact that two distinct angles in the *same quadrant* cannot have the same sine.

If the given value of s is negative, nothing essential is changed in the above argument, except that θ will then be in the third or fourth quadrant instead of being an acute angle. It will still be true that all of the angles given by formulæ (1) and (2), and no others, have the same sine as θ.

Now we may include all of the angles (1) and (2) in the single expression

$$(3) \qquad m \cdot 180° \pm \theta,$$

where m may be any integer, even or odd, and where the $+$ or $-$ sign is to be used according as m is even or odd. We may avoid this awkward distinction by writing in place of (3)

$$(4) \qquad m \cdot 180° + (-1)^m \theta,$$

since $(-1)^m$ is equal to $+1$ or -1 according as m is even or odd.

Thus, *if θ is one angle whose sine is equal to a given number s, the most general angle which has the same sine is*

$$m \cdot 180° + (-1)^m \theta,$$

where m is any positive or negative integer or zero.

A brief way of indicating this fact is given by the following equations:

$$(5) \qquad \sin[m \cdot 180° + (-1)^m \theta] = \sin \theta$$

or

$$\sin[m\pi + (-1)^m \theta] = \sin \theta,$$

where we use the first or the second form according as θ is expressed in degrees or in radians. Since the cosecant of an angle is the reciprocal of its sine, we have also

$$(6) \qquad \csc[m \cdot 180° + (-1)^m \theta] = \csc \theta$$

or

$$\csc[m\pi + (-1)^m \theta] = \csc \theta.$$

We find, by precisely similar considerations,

$$(7) \quad \begin{array}{lcl} \cos(2m \cdot 180° \pm \theta) = \cos \theta, & & \cos(2m\pi \pm \theta) = \cos \theta, \\ \sec(2m \cdot 180° \pm \theta) = \sec \theta, & \text{or} & \sec(2m\pi \pm \theta) = \sec \theta, \end{array}$$

and

$$(8) \quad \begin{array}{lcl} \tan(m \cdot 180° + \theta) = \tan \theta, & & \tan(m\pi + \theta) = \tan \theta, \\ \cot(m \cdot 180° + \theta) = \cot \theta, & \text{or} & \cot(m\pi + \theta) = \cot \theta, \end{array}$$

according as the angles are expressed in degrees or in radians.

EXERCISE LIII

Without making use of the trigonometric tables, find all of the angles which satisfy the following conditions:

1. $\sin\theta = \frac{1}{2}$. **3.** $\tan\theta = 1$. **5.** $\cot\theta = \sqrt{3}$. **7.** $\tan\theta = \infty$.

2. $\cos\theta = -\frac{1}{2}$. **4.** $\sec\theta = \sqrt{2}$. **6.** $\csc\theta = 1$. **8.** $\sin\theta = 0$.

Making use of the tables, find all of the angles which satisfy the following equations:

9. $\sin\theta = -.4721$. **11.** $\tan\theta = 1.7269$.

10. $\cos\theta = +.3216$. **12.** $\sec\theta = 2.7213$.

97. The inverse trigonometric functions. In the equation

$$x = \sin y, \tag{1}$$

we now propose to regard y, the angle or arc, as a function of x, the sine. We may express this new way of looking at the relation (1), by saying that

y is an angle whose sine is equal to x

or

(2) *y is an* **arc** *whose* **sine** *is equal to x,*

a statement which is usually written in the contracted form

$$y = \textbf{arc sin}\, x. \tag{3}$$

It should be noted that (1) and (3) are merely different ways of expressing the same relation between x and y. They differ only in one respect. In (1) y is regarded as given and x is to be found, while in (3) x is regarded as given and y is to be found. The relation between the functions (1) and (3), that of being *inverses* of each other, is of the same kind as in the more familiar case of the function $x = y^2$, which has as its inverse $y = \pm\sqrt{x}$.

In the same way we define the equation

$$y = \textbf{arc cos}\, x$$

to mean that y is an arc whose cosine is equal to x. Therefore this equation is equivalent to

$$x = \cos y.$$

Similarly, if $x = \tan y$, we write

$$\mathbf{y = arc\ tan\ x,}$$

and in the same way we define the symbols

$$\mathbf{arc\ cot\ x, \qquad arc\ sec\ x, \qquad arc\ csc\ x.}$$

Let us return to equations (1) and (3). We know that the sine of an angle is never numerically greater than 1. Consequently we can find no angle whose sine is equal to x if x is numerically greater than 1. We may express this as follows:

The function arc sin x is defined only for those values of x which are not numerically greater than 1.

If x is numerically less than unity, there exists not merely *one* angle whose sine is equal to x, but the number of such angles is unlimited. (See Art. 96.) If x is positive, one of the corresponding angles is a positive acute angle. If x is negative, the smallest corresponding *positive* angle is in the third quadrant. But in this case there is a *negative* acute angle whose sine is equal to the given negative value of x.

Let us use the symbol

$$\mathbf{Arc\ sin\ x,}$$

distinguished from arc sin x by the use of the capital letter A, *to indicate the numerically smallest angle or arc whose sine is equal to x.*

The function Arc sin x, like arc sin x, is defined only for the values of x between -1 and $+1$; that is, for those values of x for which

$$-1 \leqq x \leqq +1.$$

But for every such value of x, Arc sin x has only one value, while arc sin x has infinitely many values. For positive values of x, Arc sin x is a positive acute angle, and for negative values of x it is a negative acute angle. No value of Arc sin x ever exceeds the limits $\pm 90°$ or $\pm \frac{\pi}{2}$ radians, so that we shall always have

$$(5) \qquad -\frac{\pi}{2} \leqq \text{Arc} \sin x \leqq \frac{\pi}{2}.$$

We shall henceforth speak of Arc sin x as the ***principal value*** *of arc sin x*, and we know from equation (5) of Art. 96 that

$$\text{(6)} \qquad \text{arc} \sin x = m\pi + (-1)^m \text{ Arc} \sin x,$$

where m is any positive or negative integer or zero.

In many books the symbol arc sin x is used in the restricted sense which we have given to Arc sin x. For some purposes the distinction between the two functions arc sin x and Arc sin x is not important. But for certain other questions, a careful discussion of the principal value is the only way to avoid hopeless confusion.

This whole matter will become very clear if we make a graph of the function

$$\text{(3)} \qquad y = \text{arc} \sin x.$$

Since this equation between x and y has the same significance as

$$\text{(1)} \qquad x = \sin y,$$

we may plot the latter relation instead of (3). But we have already studied the graph of the similar equation

$$\text{(7)} \qquad y = \sin x \text{ (see Arts. 73 and 78)},$$

and clearly the graph of (1) may be obtained from that of (7) by interchanging x and y. In other words the graph of (1), or what amounts to the same thing, the graph of (3), is a sine curve placed in a vertical position. (See Fig. 125.)

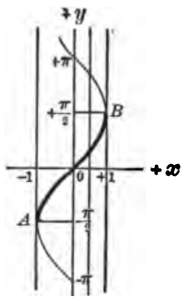

FIG. 125

The graph shows clearly that the function

$$y = \text{arc} \sin x$$

is not defined for values of x which are numerically greater than unity. It also shows that for every admissible value of x there are an infinite number of values of y, viz., the ordinates of all of the points P_1, P_2, P_3, etc., in which a line parallel to the y-axis, at a distance x, intersects the curve.

But the curve which corresponds to the principal value of arc sin x,

$$y = \text{Arc}\sin x,$$

consists merely of that portion of the graph of arc sin x which lies between the points A and B. This part of the curve is indicated in Fig. 125 by a heavier line.*

In a similar way we define a principal value Arc cos x for the function arc cos x. *The function arc cos x is defined for all values of x which are not numerically greater than* 1, *and has infinitely many values for every admissible value of x. It is convenient to define as its principal value Arc cos x, the smallest positive angle whose cosine is equal to x, so that Arc cos x is subject to the following inequalities*

$$0 \leqq \text{Arc}\cos x \leqq \pi.$$

The detailed discussion of these statements is left to the student as an exercise.

It should be remarked that the notations $\sin^{-1} x$, $\cos^{-1} x$, etc., are also in use for arc sin x, arc cos x, etc. This second notation, which is frequently used in other branches of mathematics, has the advantage of emphasizing the fact that $\sin x$ and $\sin^{-1} x$ are inverse functions of each other. But it has the disadvantage of colliding with the customary notation for exponents, and therefore tends to create confusion. Thus we usually write $\sin^2 x$ for $(\sin x)^2$, and a^{-1} for $1/a$. Thus the symbol $\sin^{-1} x$ might consistently be interpreted to mean

$$\frac{1}{\sin x} = \csc x,$$

which is something entirely different from arc sin x.

* If we had chosen as principal value of arc sin x the smallest positive angle whose sine is equal to x, the principal value would not be represented by a continuous (unbroken) curve. One part of this curve would be OB, and the other part (corresponding to negative values of x) would be between $y = \pi$ and $y = 3\pi/2$.

EXERCISE LIV

1. Show that the function $y = \text{arc} \tan x$ is defined for all values of x. Draw the graph of the function and show that its principal value may be selected subject to the conditions

$$0 \leqq \text{Arc} \tan x \leqq \pi.$$

2. Investigate the function arc $\cot x$ in the same way.

3. Are the functions arc $\sec x$ and arc $\csc x$ defined for all values of x? Draw their graphs and choose principal values for these functions.

Find the values of the following expressions:

4. Arc $\sin \frac{1}{2}$.

5. Arc $\cos \frac{1}{2}$.

6. Arc tan 1, arc tan 1.

7. Arc $\cos (\frac{1}{2}\sqrt{3})$, arc $\cos (\frac{1}{2}\sqrt{3})$.

By using the table of natural functions compute the values of the following quantities:

8. Arc tan 1.3722 + Arc cos 0.4321.

9. Arc sin 0.3425 + Arc cot 1.7264.

10. Obtain the value of $\sin (\text{Arc} \sin \frac{3}{5} + \text{Arc} \sin \frac{1}{2})$.

11. Obtain the value of $\sin (\text{Arc} \sin \frac{3}{5} + \text{Arc} \cos \frac{1}{2})$.

12. If x and y are positive numbers, less than unity, show that

$$\sin (\text{Arc} \sin x + \text{Arc} \sin y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.$$

Solution. For abbreviation put

$$\text{Arc} \sin x = u, \quad \text{Arc} \sin y = v.$$

Since x and y are positive numbers, less than unity, u and v will be positive acute angles such that

$$(1) \qquad \sin u = x, \qquad \sin v = y,$$

and therefore

$$(2) \qquad \cos u = \sqrt{1 - x^2}, \qquad \cos v = \sqrt{1 - y^2}.$$

We now find

$$\sin (\text{Arc} \sin x + \text{Arc} \sin y) = \sin (u + v) = \sin u \cos v + \cos u \sin v,$$

and therefore, on account of (1) and (2),

$$\sin (\text{Arc} \sin x + \text{Arc} \sin y) = x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.$$

13. Show that $\cos (\text{Arc} \sin x - \text{Arc} \sin y) = \sqrt{1 - x^2}\sqrt{1 - y^2} + xy$ if $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

14. Show that $\text{Arc} \sin x + \text{Arc} \cos x = \frac{\pi}{2}$ if $-1 \leqq x \leqq +1$.

15. Prove the formula Arc tan x − Arc tan y = Arc tan $\dfrac{x-y}{1+xy}$.

16. Do the equations of Exs. 12 and 13 undergo any modifications if negative values of x and y are admitted? Discuss in order the cases $x<0, y>0$; $x>0, y<0$; $x<0, y<0$.

98. Trigonometric equations. It often happens that angles are to be determined by means of equations which they must satisfy. Such equations usually contain the trigonometric functions of the unknown angle and are then known as *trigonometric equations*.

Let us confine our attention to the case where *one unknown angle* is to be found as a solution of *one equation*. Such an equation may have one of the following forms.

I. It contains θ algebraically, but does not contain the trigonometric functions of θ.

EXAMPLE. $$\theta^2 - \frac{\pi}{2} = 0.$$

II. It contains the trigonometric functions of the angle θ in algebraic combinations, but does not contain the angle θ itself explicitly.

EXAMPLES. $\sin^2\theta - \cos\theta = 0,\ \ 2\tan\theta + \cot\theta = -3.$

III. It contains the angle θ and its trigonometric functions simultaneously.

EXAMPLE. $$\theta - \frac{1}{2}\sin\theta = \frac{\pi}{2}.$$

Clearly the equations of the first type are merely algebraic equations, and their discussion properly belongs to a treatise on algebra.

It may be shown that the equations of the second type can also be reduced to algebraic equations, although it is often easier to effect their solution without so reducing them.

The equations of the third type will not be considered in this book. The solution of such equations is a difficult matter and can be accomplished in a satisfactory manner only in a few cases. It should be mentioned, however, that

the method of graphs usually provides an approximate solution for such equations.

We shall now discuss a few simple examples of equations of the second type, in such a way as to illustrate the fact that their solution may be reduced to that of algebraic equations.

EXERCISE LV

1. Solve the equation $2\cos^2 x - 5 + 7\sin x = 0$.

Solution. We have $\cos^2 x = 1 - \sin^2 x$. Therefore the given equation becomes

$$-3 - 2\sin^2 x + 7\sin x = 0.$$

Consequently, if we put $\sin x = s$, we obtain the quadratic equation for s,

$$2s^2 - 7s + 3 = 0.$$

The solution of this equation gives

$$s = 3 \text{ or } s = \tfrac{1}{2}.$$

The first solution must be discarded, since $s = \sin x$ cannot be numerically greater than unity. The second solution tells us that

$$\sin x = \tfrac{1}{2},$$

so that $x = 30°$ and $x = 150°$ are the only positive angles smaller than 360° which satisfy the equation.

In examples 2 to 5, find all positive angles less than 360° which satisfy the given equations. This may be done without using any tables.

2. $\cos 2\theta + \cos\theta = -1$.

3. $\cot 2\theta + \tan\theta = -\tfrac{2}{3}\sqrt{3}$.

4. $\tan 2\theta = -2\sin\theta$.

5. $\sec^2\theta + \csc^2\theta = 4$.

In solving the following equations, make use of the tables:

6. $\sin x \tan x = -\tfrac{9}{20}$.

7. $\cos x \cot x = -\tfrac{5}{6}$.

8. Solve the equation $\cos 3x = \sin 2x$.

99. The equation $a\sin\theta + b\cos\theta = c$. The equations of the form

$$(1) \qquad a\sin\theta + b\cos\theta = c$$

may be solved by the method of the preceding article. But, unless the numbers a, b, c are especially simple, it is far more convenient to proceed as follows:

We introduce two auxiliary quantities l and L, such that

$$(2) \qquad \begin{aligned} l\sin L &= a, \\ l\cos L &= b, \end{aligned}$$

where, moreover, l is assumed to be positive. That it is always possible to find a positive number l and an angle L which satisfy equations (2) becomes obvious if we think of a and b as the rectangular coördinates of a point P. (See Fig. 126.) Equations (2) show that l is the radius vector OP of P and that L is the angle which OP makes with the positive direction of the x-axis. Our figure also shows us that

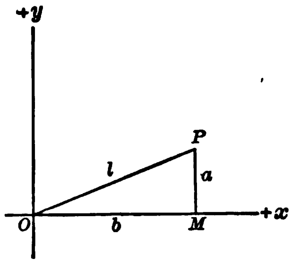

FIG. 126

(3) $$\tan L = \frac{a}{b}, \quad l = +\sqrt{a^2 + b^2}.$$

Of course these same equations also follow directly from (2) without the use of geometry.

If now we substitute the values (2) for a and b in (1), we find

$$l(\sin L \sin \theta + \cos L \cos \theta) = c,$$

or by Art. 79, equation (8),

(4) $$l \cos(\theta - L) = c.$$

Therefore, in order to solve (1) we may first determine l and L from (2) and then find θ from (4).

EXERCISE LVI

1. Solve the equation $2.1346 \sin \theta - 3.0526 \cos \theta = 0.9875$.

Solution.

$a = +\,2.1346$	(1)
$b = -\,3.0526$	(2)
$c = +\,0.9875$	(3)
$\log a = \log(l \sin L) = 0.32932$	(4)
$\log b = \log(l \cos L) = 0.48467\, n$	(5)
$\log \tan L = 9.84465\, n$	(7) = (4) − (5)
$L = 145° \; 2'.15$	(8)
$\log \sin L = 9.75820$	(9)
$\log \cos L = 9.91355\, n$	(10)
$\log l = 0.57112$	(11) = (4) − (9) = (5) − (10)
$\log c = 9.99454$	(6)
$\log \cos(\theta - L) = 9.42342$	(12) = (6) − (11)

$$\begin{aligned}
\theta_1 - L &= 74^\circ\ 37'.61 \quad (13)\\
\theta_2 - L &= 285^\circ\ 22'.39 \quad (14)\\
L &= 145^\circ\ \ \ 2'.15 \quad (8)\\
\theta_1 &= 219^\circ\ 39'.76 \quad (15) = (13) + (8)\\
\theta_2 &= 70^\circ\ 24'.54 \quad (16) = (14) + (8)
\end{aligned}$$

Remarks. The numbers in parentheses indicate the order in which the results are written down and how some of them are obtained. The -10 has not been written down in the case of negative characteristics. The n which follows the logarithms of b, $\tan L$ and $\cos L$ indicates that the corresponding numbers are negative. (See Art. 25.) L is chosen in the second quadrant because, l being positive, $\sin L$ is positive and $\cos L$ is negative. The values $\theta_1 - L$ and $\theta_2 - L$, both obtained from (12), are the two values, less than 360°, which $\theta - L$ may have so as to correspond to the given value (12) of $\log \cos(\theta - L)$,

Check. By substitution in the original equation,

$$\begin{array}{ll|ll}
\log a & = 0.32932 & \log b & = 0.48467\ n\\
\log \sin \theta_1 & = 9.80500\ n & \log \cos \theta_1 & = 9.88639\ n\\
\hline
\log(a \sin \theta_1) & = 0.13432\ n & \log(b \cos \theta_1) & = 0.37106
\end{array}$$

$$\begin{aligned}
a \sin \theta_1 &= -\ 1.3624\\
b \cos \theta_1 &= +\ 2.3500\\
\hline
a \sin \theta_1 + b \cos \theta_1 &= +\ 0.9876\\
c &= +\ 0.9875
\end{aligned}$$

This calculation checks θ_1 and therefore also L. The relation between θ_1, θ_2, and L is so simple as to make unnecessary a separate check for θ_2.

2. Solve the equation

$$-\ 3.2471 \sin \theta + 5.7469 \cos \theta = -\ 6.3271.$$

3. Solve the equation

$$2.1725 \sin \theta + 3.2749 \cos \theta = 5.7216.$$

CHAPTER XIV

APPLICATIONS TO THE THEORY OF WAVE MOTION

100. Simple harmonic motion. When we began to study trigonometry, it was for a very practical purpose. We wished to find an arithmetical method for solving triangles. We accomplished this purpose by means of the trigonometric functions and by using tables of the numerical values of these functions. Later we generalized the notion of the trigonometric functions more than was strictly necessary for the simple problem of solving triangles, and we found it to be an interesting task to investigate these trigonometric functions and their various properties for their own sake. We shall now find that these properties, aside from their theoretical interest, have in their turn most important applications.

The fundamental reason for the great importance of the trigonometric functions lies in their *periodicity*. (Cf. Art. 67.) Many natural phenomena are periodic in character, and whenever the attempt is made to represent such a phenomenon by a mathematical expression, the trigonometric functions are found to be indispensable.

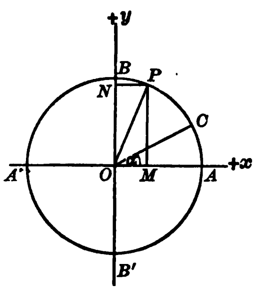

FIG. 127

The simplest periodically recurring motions are connected with uniformly rotating bodies. Let the point P (Fig. 127) describe a circular path of radius a around the point O as center, and let us assume that it is moving with a constant angular velocity of ω radians per second. Let us assume further that P starts its motion at the time $t = 0$ from the point C, which is so located that $\angle AOC$ is equal to α radians.

If P is the position of the point at the time t, that is, t seconds after the motion has begun, we shall have

(1) $$\theta = \omega t + \alpha,$$

where θ denotes the angle AOP expressed in radians. The point will describe its circular path in counterclockwise or clockwise fashion according as ω is positive or negative.

While the point P is moving in its circular path, its projection M upon the x-axis will oscillate to and fro between the points A and A', reaching its greatest speed when at O, gradually slowing down until it reaches A', when it reverses the direction of its motion, returns with gradually increasing speed to O, after reaching which point it slows down again until it reaches A and again reverses its motion.

To find an analytic expression for the motion of the point M, we observe that

$$OM = x, \quad OP = a, \quad AOP = \theta, \quad OM = OP \cos AOP,$$

whence, making use of (1),

(2) $$x = a \cos (\omega t + \alpha).$$

In the same way we see that N, the projection of P upon the y-axis, moves in accordance with the equation

(3) $$y = a \sin (\omega t + \alpha).$$

Equations (2) and (3) are so closely related that it will suffice to study one of them. In fact, if we put in (3) $\alpha = \frac{\pi}{2} + \alpha'$, it becomes (cf. Art. 77, equations (2)),

$$y = a \sin \left(\omega t + \alpha' + \frac{\pi}{2}\right) = a \cos (\omega t + \alpha'),$$

which is of the same form as (2). We shall therefore confine our attention to equation (3).

Since any diameter of the circle may be chosen as y-axis, we may express our result as follows. *If a point describes a circular path with uniform velocity, its projection upon any fixed diameter of the circle moves in accordance with an equation of the form* (3). Such a motion is called a **simple harmonic motion.** The quantity a which measures the maximum

distance of the point N from its mean position O is called the **amplitude**.

101. The period and phase constant. When the angle θ has increased from its initial value α by 2π radians, the point P will have described a complete circumference and the motion of the point N will have passed through all of its phases. The time T, which is required to accomplish this, is called the **period** of the simple harmonic motion. The period is determined by the condition that the angle ωT described by the point P in the time T must be equal to 2π. Therefore we find

$$\omega T = 2\pi, \quad T = \frac{2\pi}{\omega}. \tag{1}$$

If we wish to put the period into evidence in the equation of a simple harmonic motion, we observe that (1) gives

$$\omega = \frac{2\pi}{T},$$

so that we may write, in place of (3), Art. 100,

$$y = a \sin\left(\frac{2\pi t}{T} + \alpha\right). \tag{2}$$

This equation represents a simple harmonic motion of period T and of amplitude a. The quantity α is called the **phase constant.** The phase constant is an angle and enables us to calculate the distance from O to the position occupied by the moving point N at the time $t = 0$ when the motion began.

Thus, if $\alpha = 0$, the point N starts from O as its initial position and begins to move upward; if $\alpha = \frac{\pi}{2}$, the initial position of N is at B, etc.

Let us think of two different points oscillating up and down along the line BB', each in accordance with an equation of the form (2), the amplitudes and periods of the two motions being the same while the phase constants are different. Then these two points will move in exactly the same way, only that one will always be ahead of the other. The second point will appear to be imitating the motion of the first,

lagging behind it in a perfectly definite fashion. This is what is meant by saying that the phases of the two motions are different, the **phase** of the motion (2) at the time t being equal to

$$\frac{2\pi t}{T}+\alpha.$$

The use of the word *phase* in this connection is not merely accidental. The appearance of the moon at a given instant (its phase) depends upon the place in its orbit around the earth which it happens to occupy at that time. By analogy we speak of a periodic phenomenon as passing through all of its phases in the course of a period, and of course two otherwise identical periodic phenomena may have their corresponding phases occur at different times. It is this difference which manifests itself in the different values of the phase constant.

We have already observed that the substitution $\alpha=\frac{\pi}{2}+\alpha'$ converts (3), Art. 100, into an equation of form (2), Art. 100. We may now express this fact as follows:

The two equations

$$y_1=a\sin\frac{2\pi t}{T},\quad y_2=a\cos\frac{2\pi t}{T}=a\sin\left(\frac{2\pi t}{T}+\frac{\pi}{2}\right)$$

represent two simple harmonic motions of amplitude a and period T, which differ only in phase, the phase difference being equal to $\frac{\pi}{2}$ radians or $90°$.

The time interval which elapses between corresponding phases of these two motions is $\frac{1}{4}T$, that is, a quarter period.

102. Some illustrations of simple harmonic motion. The notion of simple harmonic motion is of fundamental importance in many problems of applied mathematics. The motion of a simple pendulum, the vibrations of a tuning fork, and many of the motions of elastic bodies may be described conveniently in terms of simple harmonic motion. The vertical motion of a particle of a water wave is approximately of the same type, and the whole theory of sound and light is based on the idea of harmonic motion.

EXERCISE LVII

In Exs. 1–5 the unit of time is one second and the unit of length one inch. Describe completely the simple harmonic motion given by each of these equations; that is, determine their amplitudes, periods, and phase constants, assuming equation (2) of Art. 101 as the standard form for the equation of such a motion.

1. $$y = 2.3745 \sin \frac{2\pi t}{5}.$$

2. $$y = 3.7216 \sin \left(4t + \frac{\pi}{12}\right).$$

3. $$y = 1.4712 \sin (2.7215t + 1.7291).$$

4. $$y = 11.7261 \cos (7t).$$

5. $$y = 2.7268 \sin \left(\frac{2\pi t}{16.7214} + 13°\right).$$

6. A certain pendulum has a period of oscillation of 5 seconds. Its motion is started by displacing the bob from its position of equilibrium three inches toward the right and then releasing it. Write the equation of its motion. What will be the corresponding equation for a point halfway between the lower end of the pendulum and its point of support?

Hint. Since the amplitude of the oscillation is small as compared with the length of the pendulum, the motion may be regarded as taking place approximately in a straight line. Take this line as x-axis, positive toward the right, and choose as origin the point of equilibrium. Let the time t be measured in seconds from the moment in which the pendulum is released. Then the required equation for the lower end of the pendulum will be

$$x = 3 \sin \left(\frac{2\pi t}{5} + \frac{\pi}{2}\right).$$

7. A cork is bobbing up and down owing to the passing water waves. These waves are 4 inches high (*i.e.* the difference of level between the crest and trough is 4 inches). If seventy of these waves pass in a minute, and we start to count time from one of the instants when the cork has reached its highest position, what equation will describe the motion of the cork approximately?

8. Show that the following mechanism enables us to convert uniform circular motion into simple harmonic motion. RR' (Fig. 128) is a rod which may slide back and forth in its own direction, its motion being limited by the guides A, B, A', B'. Attached to the rod there is a slot S perpendicular to the rod. A crank C, moving with uniform velocity in a

circle around the point O, is made to fit accurately into the slot S. Every point of the rod RR' will then describe a simple harmonic motion.

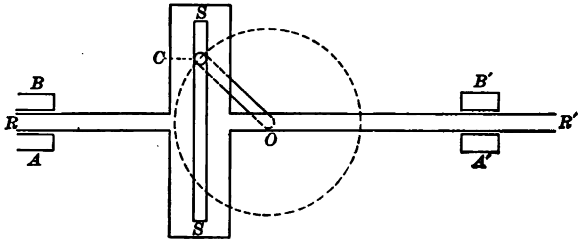

FIG. 128

9. Show that a point on the piston rod of a steam engine will move to and fro approximately according to the law of simple harmonic motion.

103. Simple harmonic curves. In the equation characteristic of a simple harmonic motion, namely,

$$y = a \sin (\omega t + \alpha),$$

let us substitute x in place of t and interpret x and y as the rectangular coördinates of a point in a plane. The curves obtained as a result of plotting such equations,

$$y = a \sin (\omega x + \alpha),$$

are called **simple harmonic curves.**

We may also think of the relation between simple harmonic motion and simple harmonic curves in the following more concrete fashion. Attach a light pin P (Fig. 129) to one of the prongs of a vibrating tuning fork, and allow it to press lightly against a strip of smoked glass. If this strip of glass is at rest, the pin will make a short straight line upon it. But if the strip be moved with a constant velocity in a direction perpendicular to that of the vibration of the tuning fork, the point P will describe a wavelike curve upon the slide. This curve may be regarded as a *record* of the motion of the tuning fork. It is easy to see that the record produced in this way by any simple harmonic motion is a simple harmonic curve.

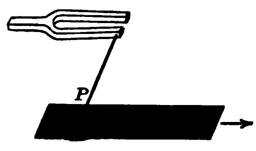

FIG. 129

The simplest case of a simple harmonic curve is that of the sine curve

$$y = \sin x,$$

whose form has, by this time, become familiar to the student. (Compare the middle curve in Fig. 130.) It has the form of a wave line with *nodes* at the points $x = 0$, $x = \pm \pi$, $x = \pm 2\pi$, etc., with *crests* or *maxima* one unit high above the points

$$x = \frac{\pi}{2},\ x = \frac{5\pi}{2},\ x = \frac{9\pi}{2},\ \text{etc.},$$

of the x-axis, and with *troughs* or *minima* one unit deep below the points

$$x = \frac{3\pi}{2},\ x = \frac{7\pi}{2},\ x = \frac{11\pi}{2},\ \text{etc.},$$

of the x-axis. The length of one complete undulation of the curve is equal to 2π.

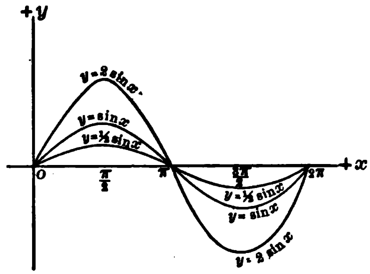

FIG. 130

104. Amplitude. It is clear that the curve

(1) $$y = a \sin x,$$

where a is any fixed positive number, is of the same general form as the sine curve. It has the same nodes (points of intersection with the x-axis), and each of its undulations has the same length as that of the sine curve. Its maxima are above the same points of the x-axis as those of the sine curve, but they are higher or lower according as a is greater or less than unity. a is called the **amplitude** of the curve. Figure 130 shows three such curves of amplitude, $\frac{1}{2}$, 1, and 2. It is clear, then, that the depth of the wavelike curve (1) is dependent upon the value of the amplitude.

105. Wave length. The two curves

$$y = \sin x \text{ and } y = \sin 2x$$

are shown together in Fig. 131. The latter curve has the same general form as the former, but each of its undulations (its wave length) is only half as long.

Similarly, the curve

(1) $$y = \sin nx,$$

where n is a positive integer, is found to be a wave line of the same height as the sine curve, but having n complete undulations between $x = 0$ and $x = 2\pi$. Consequently its wave length λ is given by

$$(2) \qquad \lambda = \frac{2\pi}{n}.$$

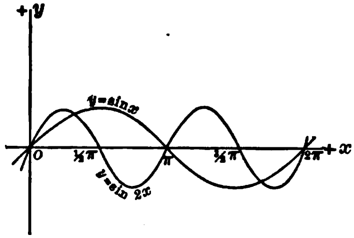

FIG. 131

Equation (1) represents a simple harmonic curve of wave-length $\lambda = \frac{2\pi}{n}$, even if n is not an integer. For the function $\sin nx$ will pass through all of its values just once, while nx changes from 0 to 2π; that is, while x changes from 0 to $\frac{2\pi}{n}$.

We may put the wave length into evidence in the equation of the curve, by solving (2) for n and substituting the resulting value of n in (1). We find $n = \frac{2\pi}{\lambda}$, and therefore

$$(3) \qquad y = \sin \frac{2\pi x}{\lambda}$$

as the equation of a simple harmonic curve of wave length λ. If we combine this result with that of Art. 104, we see that

$$(4) \qquad y = a \sin \frac{2\pi x}{\lambda}$$

represents a simple harmonic curve of wave length λ and of amplitude a.

106. Phase constant. If we compare the curves

$$y = \sin x \text{ and } y = \cos x,$$

we observe that they are identical in form. That is, both are simple harmonic curves of the same wave length and amplitude. They differ only in position. We can, in fact, slide one of these curves along the x-axis in such a way as to

make it coincide with the other. This, as we have observed before (Art. 78), is the geometrical significance of the equation

$$\sin\left(\frac{\pi}{2}+x\right)=\cos x.$$

We may therefore dispense with the cosine curve altogether and regard it as a displaced sine curve. More generally, the same thing is true of the curve

$$(1)\qquad y=\sin(x+\alpha),$$

which coincides with the sine curve if $\alpha=0$ and with the cosine curve if $\alpha=\frac{\pi}{2}$. *Equation* (1) *represents a sine curve displaced toward the left through a distance of* α *units.* The quantity α is called the **phase constant** of this curve.

If we combine the results of Arts. 104 and 105 with our latest remark, we see that

$$(2)\qquad y=a\sin\left(\frac{2\pi x}{\lambda}+\alpha\right)$$

represents a simple harmonic curve of wave length λ, *amplitude* a, *and phase constant* α.

This equation may be put into a different form by making use of the addition formula of the sine. (See Art. 79.) For we have

$$y=a\left[\sin\frac{2\pi x}{\lambda}\cos\alpha+\cos\frac{2\pi x}{\lambda}\sin\alpha\right].$$

Consequently, if we put

$$(3)\qquad m=a\cos\alpha,\quad n=a\sin\alpha,$$

we shall find

$$(4)\qquad y=m\sin\frac{2\pi x}{\lambda}+n\cos\frac{2\pi x}{\lambda}.$$

Conversely, any equation of the form (4) *has a simple harmonic curve as its graph.*

For if m and n are given, we may compute a and α from (3), thus obtaining the value of amplitude and phase constant. Since a is positive, we have, from (3),

$$(5) \qquad a = +\sqrt{m^2 + n^2}, \quad \tan \alpha = \frac{n}{m},$$

the quadrant of α being determined from the sign of its sine and cosine as given by (3).

EXERCISE LVIII

Draw the simple harmonic curves which correspond to the equations given in Exs. 1 to 5:

1. $y = 2 \sin\left(3x + \frac{\pi}{4}\right)$.

2. $y = 3 \cos\left(2x - \frac{\pi}{6}\right)$.

3. $y = 3.1 \sin(7.6x + 6.2)$.

4. $y = 2.8 \sin\left(\frac{2\pi x}{3.8} + 72°\right)$.

5. $y = \sin(x + \pi)$, $y = -\sin x$.

6. In the general theory of simple harmonic curves we assumed a to be a positive quantity. Show that this assumption does not, after all, really exclude from consideration those cases in which a is negative. In other words show that a simple harmonic curve, for which a is negative, coincides with another one for which a is positive and whose phase constant differs from that of the first curve by π radians.

7. Write the equations of the curves of Exs. 1 to 5 in the form (4) of Art. 106.

8. A simple harmonic curve is given by the equation

$$y = 2.75 \sin \frac{2\pi x}{5.76} + 3.76 \cos \frac{2\pi x}{5.76}.$$

Determine its wave length, amplitude, and phase constant.

9. Discuss in the same way the equation

$$y = 3.72 \cos(7.52x) - 2.67 \sin(7.52x).$$

107. Wave motion. Our use of the word *wave*, in connection with simple harmonic curves, is not quite in accordance with the accepted meaning of this term. Ordinarily when we speak of a wave, a water wave, for example, we mean a peculiar kind of motion. If the cross section of the surface of the water at a given instant is a simple harmonic

curve, we should properly speak of this curve, not as the wave, but as the instantaneous *profile of the wave*.* It is characteristic of a wave that this profile is in motion.

We shall obtain an excellent idea of wave motion by allowing a simple harmonic curve to glide along the x-axis with a uniform velocity v. We shall call such a wave a *simple harmonic wave*, and v its *velocity of propagation*.

Let t denote the time (expressed in seconds), and let us assume that v, the velocity of propagation (expressed in feet per second), is positive, so that the wave advances in the direction of the positive x-axis. Let the simple harmonic curve

$$(1) \qquad y = a\sin\left(\frac{2\pi x}{\lambda} + \alpha\right)$$

be the wave profile at the time t. The profile of the wave at the time $t = 0$ (t seconds earlier) was a curve of the same form as (1), but situated farther toward the left. Therefore its equation can differ from (1) only in the value of the phase constant. Consequently we may assume that

$$(2) \qquad y = a\sin\left(\frac{2\pi x}{\lambda} + \alpha_0\right)$$

is the equation of the wave profile at the time $t = 0$. We wish to find the relation between α, α_0, v, λ, and t.

The nodes of the wave profile at the time $t = 0$, that is, its intersections with the x-axis, are obtained from (2) by equating y to zero. These nodes (see Fig. 132), the points M_0, M_0', M_0'', etc., are infinite in number, and the distance between two consecutive ones is equal to $\frac{1}{2}\lambda$ or one half of the wave length. One of these nodes, M_0 say, will be obtained by equating $\frac{2\pi x}{\lambda} + \alpha_0$ to zero. Since OM_0 is the abscissa of the point M_0 and since for this point

$$\frac{2\pi x}{\lambda} + \alpha_0 = 0,$$

* The wave profile is the cross section which one would obtain of the surface of the water if it were to freeze suddenly while a wave is passing.

we shall have

$$(3) \qquad OM_0 = -\frac{\lambda\alpha_0}{2\pi}.$$

After t seconds, the wave profile has moved from its original position $M_0A_0M_0' \cdots$ to $MAM' \cdots$. The equation of the wave profile is now given by (1). The nodes of this profile are the points of the curve for which

$$\frac{2\pi x}{\lambda} + \alpha = k\pi,$$

where k is either equal to zero or to a positive or negative integer. Consequently these nodes are the points of the x-axis whose abscissas have the values

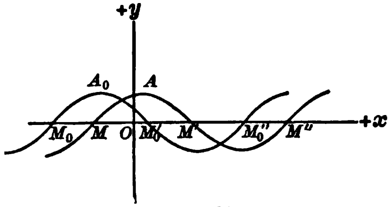

FIG. 132

$$x = -\frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda, \text{ where } k = 0, \pm 1, \pm 2, \pm 3, \cdots.$$

One of these points is the new position M occupied by the point M_0 as a consequence of the motion of the wave profile from $M_0A_0M_0' \cdots$ to $MAM' \cdots$. Since OM is the abscissa of M, we shall therefore have

$$(4) \qquad OM = -\frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda,$$

where k is a definite positive or negative integer or zero whose precise value remains to be determined.

But if v is the velocity of propagation of the wave in feet per second, every point on it moves through a distance of vt feet in t seconds. Therefore

$$(5) \qquad OM - OM_0 = vt.$$

If we substitute in this equation the values (3) and (4) for OM and OM_0, we find

$$(6) \qquad \frac{\lambda\alpha_0}{2\pi} - \frac{\lambda\alpha}{2\pi} + \tfrac{1}{2}k\lambda = vt.$$

This equation must be true for all values of t. In particular it must be true for $t = 0$. But for $t = 0$ we have $\alpha = \alpha_0$, so that the equation involves a contradiction unless $k = 0$. Consequently the integer k which appears in equations (4) and (6) must be equal to zero, and we have

$$\frac{\lambda\alpha_0}{2\pi} - \frac{\lambda\alpha}{2\pi} = vt, \text{ or } \frac{\lambda}{2\pi}(\alpha_0 - \alpha) = vt;$$

whence

$$\alpha = \alpha_0 - \frac{2\pi vt}{\lambda}. \tag{7}$$

If we substitute this value of α in (1), we find

$$y = a \sin\left[\frac{2\pi}{\lambda}(x - vt) + \alpha_0\right] \tag{8}$$

as the general equation of a simple harmonic wave of amplitude a and wave length λ, whose velocity of propagation is equal to v. We may still speak of α_0 as the phase constant. It is the phase constant of the wave profile at the time $t = 0$. The phase constant of the profile curve at any other instant may be computed from (7).

If in (8) we assign a fixed value to t, we obtain the equation of the wave profile at that instant. Let us instead assign a fixed value to x, so that y becomes a function of t alone. In the case of a water wave this would amount to a study of the upward and downward oscillations of a cork floating upon the water. We shall naturally inquire as to the length of time which is required to complete such an oscillation. This time is called the *period* and may be denoted by T. Clearly T is the time which must be added to t so as to change the argument of the sine function in (8) by $\pm 2\pi$. Now if we increase t by T without changing x, this argument changes by $-\frac{2\pi vT}{\lambda}$, and this will be equal to -2π if

$$\frac{vT}{\lambda} = 1, \text{ or } T = \frac{\lambda}{v}.$$

Therefore, *if v is the velocity of propagation, λ the wave length, and T the period, we have the relation*

$$(9) \qquad \lambda = Tv.$$

Instead of (8) we may now write

$$(10) \qquad y = a \sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right) + \alpha_0\right]$$

as the general equation of a wave of length λ, period T, amplitude a, and phase constant α_0.

We see finally that *a simple harmonic wave has for its profile at any moment a simple harmonic curve, and that every point upon it oscillates up and down in accordance with the law of simple harmonic motion.*

EXERCISE LIX

If the units of length and time are feet and seconds respectively, compute the wave length, period, amplitude, and phase constant of each of the waves represented by the following equations:

1. $y = 3 \sin\left[2\pi\left(\frac{x}{5} - \frac{t}{15}\right) + \frac{\pi}{2}\right].$

2. $y = 2 \sin\left(3x - 7t + \frac{\pi}{4}\right).$

3. $y = a \sin(bx + ct + d).$

108. General harmonic motion. A simple harmonic motion is the simplest case of an oscillatory phenomenon, and many natural periodic motions may be adequately described as simple harmonic motions. We have already given some examples of such cases. But very frequently the motion, although of an oscillatory character, is more complicated. In the case of a water wave, for instance, we see that the profile of the wave is not a simple harmonic curve, but that there are smaller waves (ripples so to speak) running along the backs of the larger ones, thus complicating the motion. Simple experiments show that the sound waves produced by a tuning fork are very approximately represented by simple harmonic motion; but other musical instruments, such as the

violin, the piano, the human voice, produce sound waves which resemble the more complicated water waves.

A tuning fork which makes 129 oscillations in a second causes a certain simple harmonic motion of the air particles whose period is $\frac{1}{129}$th of a second and which produces a certain tone usually denoted by C. If the same note is struck on the piano, it is found that the principal part of the motion of the air particles again has $\frac{1}{129}$th of a second as its period. But the motion is not simply harmonic. It is a combination of this fundamental motion with one twice as fast, with another three times as fast, and so on. In other words, the motion of the air particles is given by an equation of the form

$$(1)\quad y = a_1 \sin\left(\frac{2\pi}{T}t + \alpha_1\right) + a_2 \sin\left(\frac{4\pi}{T}t + \alpha_2\right) + a_3 \sin\left(\frac{6\pi}{T}t + \alpha_3\right) + \cdots,$$

where the period of the first and principal term is T, that of the second $\frac{1}{2}T$, that of the third $\frac{1}{3}T$, etc.

This is not the place to discuss details of the theory of sound. Our purpose in entering upon this theory at all was merely to explain one of the many instances in which sums of simple harmonic functions of the form (1) present themselves as indispensable.

We wish to learn how the various terms in (1) combine. For that purpose the length of the period T makes but little difference. We shall therefore put

$$T = 2\pi$$

since the formulæ will then assume a somewhat simpler appearance. Then (1) reduces to

$$(2)\quad y = a_1 \sin(t + \alpha_1) + a_2 \sin(2t + \alpha_2) + a_3 \sin(3t + \alpha_3) + \cdots.$$

Now each of these terms may be expanded in accordance with the addition theorem (Art. 79), so that

$$a_1 \sin(t + \alpha_1) = a_1 \cos\alpha_1 \sin t + a_1 \sin\alpha_1 \cos t,$$
$$a_2 \sin(2t + \alpha_2) = a_2 \cos\alpha_2 \sin 2t + a_2 \sin\alpha_2 \cos 2t, \text{ etc.}$$

Consequently, if we introduce new constants $A_1, A_2, \cdots, B_1, B_2, \cdots$ by putting

$$A_1 = a_1 \sin \alpha_1, \quad A_2 = a_2 \sin \alpha_2, \cdots,$$
$$B_1 = a_1 \cos \alpha_1, \quad B_2 = a_2 \cos \alpha_2, \cdots,$$

equation (2) becomes

$$(3) \qquad \begin{aligned} y = A_1 \cos t + A_2 \cos 2\,t + A_3 \cos 3\,t + \cdots \\ + B_1 \sin t + B_2 \sin 2\,t + B_3 \sin 3\,t + \cdots. \end{aligned}$$

Let us call the fixed point, with which the moving point would tend to coincide if the amplitudes a_1, a_2, etc., of all the simple harmonic motions of (1) were to approach the limit zero, the *center of oscillation.* We have tacitly assumed so far that the center of oscillation was the origin of coördinates. Let us drop this specialization, and let $\frac{1}{2} A_0$ be the fixed value to which y would reduce if all of the oscillations were to disappear; that is, let $\frac{1}{2} A_0$ be the ordinate of the center of oscillation.* Then we must add $\frac{1}{2} A_0$ to the right member of (3), so that we obtain finally

$$(4) \qquad \begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos t + A_2 \cos 2\,t + A_3 \cos 3\,t + \cdots \\ + B_1 \sin t + B_2 \sin 2\,t + B_3 \sin 3\,t + \cdots \end{aligned}$$

as the typical equation of a general harmonic motion.

As in the case of a *simple* harmonic motion, we may make a graphical *record* of this motion. Suppose, for instance, that the motion to be investigated is the vibration of a metallic wire. We attach a light pen to the wire so as to enable it to write upon a strip of smoked glass. If the glass be left at rest while the wire is caused to vibrate, the pen will merely describe a straight line upon the smoked glass. If, however, the glass be moved with a rapid uniform motion at right angles to the direction of vibration of the wire, there will appear as record a wavelike curve. This curve will belong to the class considered in the next article.

109. General harmonic curves. If we put x in place of t in equation (4) of Art. 108, we find

$$(1) \qquad \begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2\,x + A_3 \cos 3\,x + \cdots \\ + B_1 \sin x + B_2 \sin 2\,x + B_3 \sin 3\,x + \cdots. \end{aligned}$$

* The reason for denoting this quantity by $\frac{1}{2} A_0$ rather than A_0 will appear later. (See Art. 112.)

The curves which are obtained as a result of plotting an equation of this form are called *general harmonic curves* and are capable of an extraordinary variety of forms. In fact it can be shown, by methods involving the integral calculus, that an *infinite* series of the form (1) may be found to represent almost *any* continuous curve, and even extensive classes of discontinuous curves*. In this book, however, we are concerned only with sums of the form (1) involving a *finite* number of terms and the curves represented by them. The name *harmonic curves* will be understood to apply only to such curves.

We proceed to discuss an example. Let us plot the curve whose equation is

$$(2) \qquad y = \sin x + \sin 2x.$$

We begin by drawing the two familiar curves

$$(3) \qquad y_1 = \sin x \text{ and } y_2 = \sin 2x,$$

the two dotted curves of Fig. 133. From these curves it is easy to construct the curve (2). For we see from (2) and (3) that

$$y = y_1 + y_2$$

for every value of x. If then we find the ordinate of each of the two dotted curves for a given value of x, their algebraic sum will be the ordinate of a point on the required curve. The resulting curve is indicated in Fig. 133 by a full line. A few points of this curve may easily

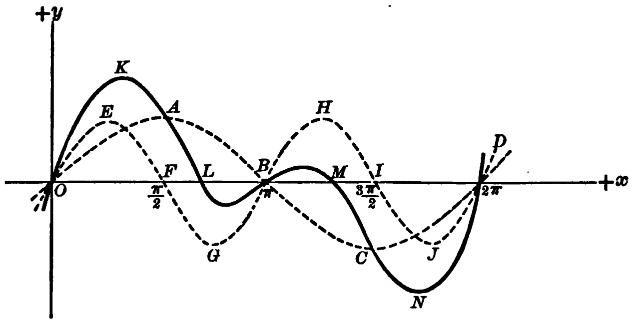

FIG. 133

* Such series are usually called FOURIER'S series in honor of the great mathematical physicist who first stated, and in part proved, the above theorem. The first rigorous proof was furnished much later by DIRICHLET.

be obtained by inspection. For $x = 0$, y_1 and y_2 are both zero, and therefore also $y = y_1 + y_2 = 0$. Consequently the point O is on the curve. For $x = \pi/2$, $y_1 = 1$ and $y_2 = 0$, so that $y = 1$ and the curve passes through the point A. For values of x between $\pi/2$ and π, y_2 is negative so that $y_1 + y_2$ will be less than y_1. Consequently the full line curve in this interval lies below the corresponding portion AB of the curve $y_1 = \sin x$. For a certain value of x in this interval (determined by the equation $\sin x + \sin 2x = 0$) y_1 and y_2 will be numerically equal but opposite in sign, so that at that point the curve $y = \sin x + \sin 2x$ will cross the x-axis. This is the point L of Fig. 133. For any value of x we may obtain y_1 and y_2 by measurement from the two dotted curves. If we form the sum of these two quantities with due regard to sign, we find the corresponding ordinate of the required curve.*

EXERCISE LX

Plot the following harmonic curves:

1. $y = 2 \sin x + \sin 2x$.
2. $y = \sin x + \frac{1}{2} \sin 2x$.
3. $y = \sin x + \cos 2x$.
4. $y = 5 \sin x + \sin 4x$.

110. Harmonic analysis or trigonometric interpolation. We have seen how a number of simple harmonic curves may be compounded into a single general harmonic curve. It often happens that a curve is given, actually drawn out on paper, as for instance in the case of a self-recording barometer or thermometer. If the curve is of a periodic character, the question arises whether it may be regarded as a harmonic curve, that is, whether it is possible to compound it out of a number of simple harmonic curves by the method of Art. 109. And if so, the problem presents itself to actually find the component simple harmonic curves. The process of solving this problem is known as *harmonic analysis* and is of great importance in many branches of pure and applied mathematics.

Let us suppose that the given curve is periodic, so that it

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* MICHELSON and STRATTON have devised a machine for performing mechanically the operation of combining a number of simple harmonic curves. This machine is also capable of performing the inverse operation discussed in Art. 110. For this reason it has been called a *harmonic analyzer*.

consists of an infinite number of equal pieces, and let the length of one of these pieces, the wave length, be equal to λ. If λ is less than 2π, we may, by a process of magnification or stretching, replace the given curve by another one similar to it whose wave length is just equal to 2π. If we can solve our problem for this second curve of wave length 2π, it will be easy to solve the corresponding problem for the original curve. If λ is greater than 2π, a process of compression enables us to reduce the problem again to the case of a curve of wave length 2π. We may therefore assume that the wave length of the given periodic curve is equal to 2π without reducing, by this assumption, in any essential fashion the general applicability of our results. The object of this assumption is merely to simplify the resulting formulæ.

Let $P_0P_1P_2 \cdots$ (Fig. 134) be the given curve of wave length $OA = 2\pi$, and let us divide OA into an odd number,

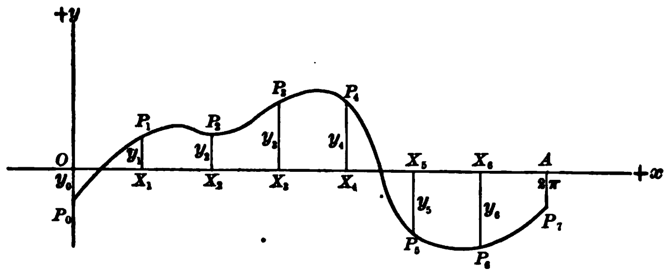

FIG. 134

say $2m + 1$, of equal parts. Not counting A, there will then be $2m + 1$ points of division; namely, O, X_1, X_2, $\cdots X_{2m}$. In Fig. 134 we have made $2m + 1 = 7$.

At these points of division we construct the ordinates

$$OP_0, X_1P_1, X_2P_2, \cdots$$

of the curve. Let $y_0, y_1, y_2, \cdots y_{2m}$ be these ordinates, each with its proper sign prefixed. On account of the periodic character of the curve, the ordinate at A will be the same as that at O. This is the reason that we did not count A as

one of the points of division. If we had included A, we should really have been counting O twice.

We can always find a harmonic curve involving terms in x, $2x$, $3x$, $\cdots mx$, which passes through the $2m+1$ points P_0, P_1, P_2, $\cdots P_{2m}$. For the general equation of such a harmonic curve is

$$(1) \qquad \begin{aligned} y = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx \\ + B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx, \end{aligned}$$

and therefore contains $2m+1$ coefficients A_0, A_1, $\cdots A_m$, B_1, $\cdots B_m$, which may be determined in such a way as to make the corresponding curve (1) pass through the $2m+1$ given points. In fact, the curve (1) will pass through the point P_0, if the value of y obtained from (1), for $x = 0$, is equal to the ordinate y_0 of the given point P_0; that is, if

$$(2) \qquad y_0 = \tfrac{1}{2}A_0 + A_1 + A_2 + \cdots + A_m.$$

The abscissa of P_1 is $OX_1 = \dfrac{2\pi}{2m+1}$. Therefore the curve (1) will pass through P_1, if the value of y obtained from (1), for $x = \dfrac{2\pi}{2m+1}$, is equal to the ordinate y_1 of the given point P_1; that is, if

$$(3) \qquad \begin{aligned} y_1 = \tfrac{1}{2}A_0 + A_1 \cos\frac{2\pi}{2m+1} + A_2 \cos\frac{2\cdot 2\pi}{2m+1} \\ + \cdots + A_m \cos\frac{m\cdot 2\pi}{2m+1} \\ + B_1 \sin\frac{2\pi}{2m+1} + B_2 \sin\frac{2\cdot 2\pi}{2m+1} \\ + \cdots + B_m \sin\frac{m\cdot 2\pi}{2m+1}. \end{aligned}$$

In the same way we find that the curve (1) will pass through the points P_2, P_3, $\cdots P_{2m}$ if the following additional equations are satisfied;

$$(4)\quad \begin{cases} y_2 = \frac{1}{2}A_0 + A_1\cos\dfrac{4\pi}{2m+1} + A_2\cos\dfrac{2\cdot 4\pi}{2m+1} + \cdots \\ \qquad + A_m\cos\dfrac{m\cdot 4\pi}{2m+1} + B_1\sin\dfrac{4\pi}{2m+1} + B_2\sin\dfrac{2\cdot 4\pi}{2m+1} \\ \qquad\qquad + \cdots + B_m\sin\dfrac{m\cdot 4\pi}{2m+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_{2m} = \frac{1}{2}A_0 + A_1\cos\dfrac{2m\pi}{2m+1} + A_2\cos\dfrac{2\cdot 2m\pi}{2m+1} + \cdots \\ \qquad + A_m\cos\dfrac{m\cdot 2m\pi}{2m+1} + B_1\sin\dfrac{2m\pi}{2m+1} + B_2\sin\dfrac{2\cdot 2m\pi}{2m+1} \\ \qquad\qquad + \cdots + B_m\sin\dfrac{m\cdot 2m\pi}{2m+1}. \end{cases}$$

Now the equations (2), (3), and (4) are together $2m+1$ in number, and involve $2m+1$ unknown quantities, viz., the quantities, $A_0, A_1, \cdots A_m, B_1, \cdots B_m$. The coefficients of these quantities (sines and cosines of known angles) are known numbers, and the quantities $y_0, y_1, \cdots y_{2m}$, the ordinates of the given points $P_0, P_1, \cdots P_{2m}$, are known. We may therefore, in general, solve these $2m+1$ equations for the $2m+1$ unknown quantities and then substitute the values obtained for $A_0, A_1, \cdots A_m, B_1, \cdots B_m$ in (1). The harmonic curve obtained by plotting (1) will then pass through the given points.

If the given curve is fairly smooth, and if the points P_0, $P_1, \cdots P_{2m}$ be taken close enough together, that is, if m be chosen large enough, we may find by the above method a harmonic curve which may be regarded as replacing the original curve everywhere with any desired degree of approximation.*

The following example will illustrate this method. Let y be a periodic function of x, of period 2π, such that the values of y which correspond to

$$x = 0°,\ 72°,\ 144°,\ 216°,\ 288°$$

* A more precise formulation of this statement must be left for a much later stage of the student's mathematical career.

are respectively

$$y = 0,\ +2,\ +\tfrac{1}{2},\ -\tfrac{1}{2},\ -2.$$

Our general theory tells us that we may find an expression of the form

$$y = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + B_1 \sin x + B_2 \sin 2x,$$

which assumes the five values assigned to y for the five given values of x. Moreover we have the following five equations for the five unknown coefficients A_0, A_1, A_2, B_1, B_2:

$$\begin{cases} 0 = \tfrac{1}{2}A_0 + A_1 + A_2, \\ 2 = \tfrac{1}{2}A_0 + A_1 \cos 72^\circ + A_2 \cos 144^\circ + B_1 \sin 72^\circ + B_2 \sin 144^\circ, \\ \tfrac{1}{2} = \tfrac{1}{2}A_0 + A_1 \cos 144^\circ + A_2 \cos 288^\circ + B_1 \sin 144^\circ + B_2 \sin 288^\circ, \\ -\tfrac{1}{2} = \tfrac{1}{2}A_0 + A_1 \cos 216^\circ + A_2 \cos 72^\circ + B_1 \sin 216^\circ + B_2 \sin 72^\circ, \\ -2 = \tfrac{1}{2}A_0 + A_1 \cos 288^\circ + A_2 \cos 216^\circ + B_1 \sin 288^\circ + B_2 \sin 216^\circ. \end{cases}$$

Since we have

$$\begin{aligned} \sin 216^\circ &= \sin(360^\circ - 144^\circ) = -\sin 144^\circ, \\ \cos 216^\circ &= \cos(360^\circ - 144^\circ) = \cos 144^\circ, \\ \sin 288^\circ &= \sin(360^\circ - 72^\circ) = -\sin 72^\circ, \\ \cos 288^\circ &= \cos(360^\circ - 72^\circ) = \cos 72^\circ, \end{aligned}$$

the above equations may also be written as follows:

(1) $\tfrac{1}{2}A_0 + A_1 + A_2 = 0,$
(2) $\tfrac{1}{2}A_0 + A_1 \cos 72^\circ + A_2 \cos 144 + B_1 \sin 72^\circ + B_2 \sin 144^\circ = 2,$
(3) $\tfrac{1}{2}A_0 + A_1 \cos 144^\circ + A_2 \cos 72^\circ + B_1 \sin 144^\circ - B_2 \sin 72^\circ = \tfrac{1}{2},$
(4) $\tfrac{1}{2}A_0 + A_1 \cos 144^\circ + A_2 \cos 72^\circ - B_1 \sin 144^\circ + B_2 \sin 72^\circ = -\tfrac{1}{2},$
(5) $\tfrac{1}{2}A_0 + A_1 \cos 72^\circ + A_2 \cos 144^\circ - B_1 \sin 72^\circ - B_2 \sin 144^\circ = -2.$

From (2) and (5) we find by addition

(6) $$\tfrac{1}{2}A_0 + A_1 \cos 72^\circ + A_2 \cos 144^\circ = 0,$$

and similarly from (3) and (4),

(7) $$\tfrac{1}{2}A_0 + A_1 \cos 144^\circ + A_2 \cos 72^\circ = 0.$$

From (1) we have

$$\tfrac{1}{2}A_0 = -A_1 - A_2$$

which, substituted in (6) and (5), gives

(8) $$\begin{aligned} A_1(\cos 72^\circ - 1) + A_2(\cos 144^\circ - 1) &= 0, \\ A_1(\cos 144^\circ - 1) + A_2(\cos 72^\circ - 1) &= 0. \end{aligned}$$

If we multiply both members of the first of these equations by $\cos 72^\circ - 1$, those of the second by $-(\cos 144^\circ - 1)$, and add, we find

(9) $$A_1[(\cos 72^\circ - 1)^2 - (\cos 144^\circ - 1)^2] = 0.$$

From the table of natural functions, we find to two decimal places

$$\cos 72^\circ = 0.31,\ \cos 144^\circ = -\cos 36^\circ = -0.81.$$

Therefore

$$\cos 72^\circ - 1 = -0.69,\ \cos 144^\circ - 1 = -1.81,$$

so that the coefficient of A_1 in (9) is not equal to zero. Consequently we conclude from (9) that $A_1 = 0$. According to (8) and (1), we must then have also $A_2 = 0$, $A_0 = 0$.

If now we put $A_0 = A_1 = A_2 = 0$, in (1) to (5), these five equations reduce to the following two:

$$\text{(10)} \qquad \begin{aligned} B_1 \sin 72^\circ + B_2 \sin 144^\circ &= 2, \\ B_1 \sin 144^\circ - B_2 \sin 72^\circ &= \tfrac{1}{2}. \end{aligned}$$

From these equations we eliminate first B_2 and then B_1, giving

$$\text{(11)} \qquad \begin{aligned} (\sin^2 72^\circ + \sin^2 144^\circ) B_1 &= 2 \sin 72^\circ + \tfrac{1}{2} \sin 144^\circ, \\ (\sin^2 72^\circ + \sin^2 144^\circ) B_2 &= 2 \sin 144^\circ - \tfrac{1}{2} \sin 72^\circ. \end{aligned}$$

From the table of natural sines we find, correct to two decimal places,

$$\sin 72^\circ = 0.95,\ \sin 144^\circ = \sin 36^\circ = 0.59,$$

so that

$$\sin^2 72^\circ + \sin^2 144^\circ = 0.90 + 0.35 = 1.25.$$

Consequently, equations (11) become

$$1.25\, B_1 = 1.90 + 0.30 = 2.20,$$

$$1.25\, B_2 = 1.18 - 0.48 = 0.70;$$

whence finally

$$\text{(12)} \qquad B_1 = 1.76,\ B_2 = 0.56.$$

Since we have already found $A_0 = A_1 = A_2 = 0$, the function which we were seeking is

$$\text{(13)} \qquad y = 1.76 \sin x + 0.56 \sin 2x.$$

In order to check our result we may substitute the five given values of x in (13) and verify that the corresponding values of y are actually those which were originally given. That equation (13) gives $y = 0$ for $x = 0$ is obvious. For $x = 72^\circ$ and for $x = 144^\circ$, we find from (13)

$$y = 1.76 \times 0.95 + 0.56 \times 0.59 = 1.67 + 0.33 = 2.00,$$

and

$$y = 1.76 \times 0.59 + 0.56 \times (-0.95) = 1.04 - 0.53 = 0.51,$$

respectively, checking to within one unit of the last decimal place computed. To check the other two pairs of values requires no additional computation.

In this example, the values of y for $x = 144^\circ = 180^\circ - 36^\circ$ and for $x = 216^\circ = 180^\circ + 36^\circ$ were numerically equal but opposite in sign. The values of y for $x = 72^\circ = 180^\circ - 108^\circ$ and for $x = 288^\circ = 180^\circ + 108^\circ$ were also numerically equal

but opposite in sign. It is owing to this circumstance that A_0, A_1, and A_2 turned out to be all three equal to zero. Having become aware of this fact, we may abbreviate our work very much by at once equating A_0, A_1, A_2, etc., to zero, whenever the given values of the function are numerically equal but opposite in sign for those among the given angles which differ numerically by the same amount from 180°.

Similarly, if the given values of the function y are equal numerically *and* in sign for all of those among the given angles x which differ numerically by the same amount from 180°, the expression for y will contain no sine terms; that is, B_1, B_2, B_3, etc., will all be equal to zero.

EXERCISE LXI

1. Find a periodic function, involving only the angles x and $2x$, which assumes the values

$$y = 0, +2, +1, -1, -2,$$

for

$$x = 0°, 72°, 144°, 216°, 288°,$$

respectively. Compute the coefficients to two decimal places.

2. Find a periodic function, involving only the angles x and $2x$, which assumes the values

$$y = +2, +1, -\tfrac{1}{2}, -\tfrac{1}{2}, +1,$$

for

$$x = 0°, 72°, 144°, 216°, 288°,$$

respectively. Compute the coefficients to two decimal places.

111. Theorems leading to the general solution of the problem of trigonometric interpolation. We have shown in Art. 110 that the problem of trigonometric interpolation may be reduced to that of solving a system of $2m+1$ equations of the first degree with $2m+1$ unknowns. But we can accomplish much more than this. We shall derive elegant and convenient formulæ for the solutions of these equations, enabling us to find the values of the coefficients A_k and B_k by a direct and simple process. But before we can do this, we must prepare the way by proving some theorems necessary for this purpose.

We begin by proving that the following formula

(1) $\sin a + \sin(a+t) + \sin(a+2t) + \cdots + \sin(a+mt)$

$$= \frac{\sin\left(a+\frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}},$$

published by EULER* in 1743, is true for all values of a and t, provided that $\sin\frac{t}{2}$ is not equal to zero.

PROOF. Let us denote the sum in the left-hand member of (1) by s_m. If we multiply s_m by $2\sin\frac{t}{2}$, we shall have

$$2s_m\sin\frac{t}{2} = 2\sin a\sin\frac{t}{2} + 2\sin(a+t)\sin\frac{t}{2}$$
$$+ 2\sin(a+2t)\sin\frac{t}{2} + \cdots + 2\sin(a+mt)\sin\frac{t}{2}.$$

Every term in the right member of this equation contains a product of two sines, and may therefore be expressed as a difference of two cosines by means of formula (4) of Art. 82; that is,

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)].$$

We find, in this way,

$$\begin{aligned} 2s_m\sin\frac{t}{2} = {} & \cos(a-\tfrac{1}{2}t) - \cos(a+\tfrac{1}{2}t) \\ & + \cos(a+\tfrac{1}{2}t) - \cos(a+\tfrac{3}{2}t) \\ & + \cos(a+\tfrac{3}{2}t) - \cos(a+\tfrac{5}{2}t) \\ & + \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \cos\{a+(m-\tfrac{1}{2})t\} - \cos\{a+(m+\tfrac{1}{2})t\}. \end{aligned}$$

* EULER (1707–1783) was born in Switzerland, but spent most of the years of his scientific career in St. Petersburg and Berlin. His work was of fundamental importance in all parts of pure and applied mathematics. Although absolutely blind during the latter part of his life, he continued to labor and to make important contributions up to the end.

Clearly all of the terms in the right member except the first and last will destroy each other, so that we are left with the equation

$$2\,s_m \sin\frac{t}{2} = \cos(a - \tfrac{1}{2}t) - \cos\{a + (m + \tfrac{1}{2})t\}.$$

But we have the formula (see Art. 82, equations (5)),

$$\cos A - \cos B = -2\sin\frac{A-B}{2}\cos\frac{A+B}{2},$$

so that

$$2\,s_m \sin\frac{t}{2} = -2\sin\frac{1}{2}(-t - mt)\sin\frac{1}{2}(2a + mt)$$

$$= 2\sin\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2},$$

whence, if $\sin\frac{t}{2}$ is not equal to zero,

$$s_m = \frac{\sin\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}}.$$

This is the same as equation (1), the formula which we wished to prove.

Let us rewrite formula (1) with a' in place of a, and then put

$$a' = a + \frac{\pi}{2}.$$

Since
$$\sin a' = \sin\left(a + \frac{\pi}{2}\right) = \cos a,$$

we then find

$$(2)\quad c_m = \cos a + \cos(a + t) + \cos(a + 2t) + \cdots + \cos(a + mt)$$

$$= \frac{\cos\left(a + \frac{mt}{2}\right)\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}},$$

a formula which may also be obtained directly by a process strictly analogous to the one employed for the proof of equation (1). Formula (2) is also due to EULER.

If in equations (1) and (2) we put $a=0$, we find

$$(3)\quad \sin t+\sin 2t+\sin 3t+\cdots+\sin mt=\frac{\sin\frac{mt}{2}\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}}$$

and

$$(4)\quad 1+\cos t+\cos 2t+\cos 3t+\cdots+\cos mt=\frac{\cos\frac{mt}{2}\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}}.$$

Let us subtract $\frac{1}{2}$ from both members of (4). We obtain the formula

$$\frac{1}{2}+\cos t+\cos 2t+\cdots+\cos mt=\frac{\cos\frac{mt}{2}\sin\frac{(m+1)t}{2}}{\sin\frac{t}{2}}-\frac{1}{2}$$

$$=\frac{2\cos\frac{mt}{2}\sin\frac{(m+1)t}{2}-\sin\frac{t}{2}}{2\sin\frac{t}{2}}=\frac{\sin\left(m+\frac{1}{2}\right)t+\sin\frac{t}{2}-\sin\frac{t}{2}}{2\sin\frac{t}{2}},$$

where we have made use of one of the equations (4) of Art. 82. The numerator of the last fraction obviously reduces to its first term, so that we find finally

$$(5)\quad \frac{1}{2}+\cos t+\cos 2t+\cos 3t+\cdots+\cos mt=\frac{\sin\frac{(2m+1)t}{2}}{2\sin\frac{t}{2}},$$

a formula which was known to SNELLIUS in 1627.

The angle t which occurs in all of these equations may have any value whatever excepting only those values for which $\sin\frac{t}{2}$ is equal to zero; that is, excepting the values $2k\pi$, where k is an integer. We shall now apply these formulæ

to the particular case when t is a commensurable fractional part of the entire circumference, so that

$$t = \frac{k}{n} 2\pi,$$

where both k and n are positive integers and where

$$n > 1.$$

We shall further put $\qquad m = n - 1.$

We may express these assumptions more concretely as follows. Let us divide the circumference of the circle into n equal parts, where $n > 1$. Then $\frac{2\pi}{n}$ will be the angle subtended by one of these parts. The smallest multiple of this angle which is equal to a complete circumference is of course the nth. Therefore the angles

$$\frac{2\pi}{n},\ 2\frac{2\pi}{n},\ 3\frac{2\pi}{n}, \cdots (n-1)\frac{2\pi}{n}$$

are all distinct, and we propose to find the sum of their sines as well as of their cosines. We consider next the angle $2\frac{2\pi}{n} = \frac{4\pi}{n}$ and all of its multiples up to $(n-1)\frac{4\pi}{n}$ and calculate the sum of their sines and of their cosines. In general, we consider the angle $k\frac{2\pi}{n} = \frac{2k\pi}{n}$ and its multiples $2\frac{2k\pi}{n}$, $3\frac{2k\pi}{n} \cdots (n-1)\frac{2k\pi}{n}$ and compute the sum of their sines and of their cosines.

According to (3) we have (putting $t = \frac{2k\pi}{n}$ and $m = n-1$)

$$\sin\frac{2k\pi}{n} + \sin 2\frac{2k\pi}{n} + \sin 3\frac{2k\pi}{n} + \cdots + \sin(n-1)\frac{2k\pi}{n}$$
$$= \frac{\sin\frac{(n-1)k\pi}{n}\sin k\pi}{\sin\frac{k\pi}{n}},$$

a formula which will be valid unless $\sin\frac{k\pi}{n}$ is equal to zero;

that is, unless k is an integral multiple of n. Excluding this case for a moment, we see that the right member is equal to zero since $\sin k\pi$ occurs as a factor in the numerator and since the sine of any integral multiple of π is equal to zero. Consider now the excluded case when k is a multiple of n. Then every term on the left member is individually equal to zero. We see therefore that

$$(6)\quad S_k = \sin\frac{2k\pi}{n} + \sin 2\frac{2k\pi}{n} + \sin 3\frac{2k\pi}{n} + \cdots + \sin(n-1)\frac{2k\pi}{n} = 0,$$

for every value of k, whether k is divisible by n or not.

We may prove in the same way, making use of equation (4), that

$$(7)\quad C_k = 1 + \cos\frac{2k\pi}{n} + \cos 2\frac{2k\pi}{n} + \cos 3\frac{2k\pi}{n} + \cdots + \cos(n-1)\frac{2k\pi}{n} = 0,$$

if k is not a multiple of n.

If k is a multiple of n, all of the angles $\frac{2k\pi}{n}$, $2\frac{2k\pi}{n}$, etc., are integral multiples of 2π, so that each of the cosines which appears in C_k is equal to unity. There are n terms in C_k. Therefore

$$(8)\quad C_k = 1 + \cos\frac{2k\pi}{n} + \cos 2\frac{2k\pi}{n} + \cdots + \cos(n-1)\frac{2k\pi}{n} = n,$$

if k is a multiple of n.

Let us now consider two angles of the form $\frac{2k\pi}{n}$ and $\frac{2l\pi}{n}$, and denote by C_{kl} the sum

$$(9)\quad C_{kl} = 1 + \cos\frac{2k\pi}{n}\cos\frac{2l\pi}{n} + \cos 2\frac{2k\pi}{n}\cos 2\frac{2l\pi}{n} + \cdots + \cos(n-1)\frac{2k\pi}{n}\cos(n-1)\frac{2l\pi}{n}.$$

A product of two cosines may be expressed as a sum by means of formula (3) of Art. 82; that is,

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)].$$

Therefore we may write, if we apply this formula to each term of C_{kl},

$$\begin{aligned} C_{kl} &= \tfrac{1}{2}[1+1] \\ &+ \tfrac{1}{2}\left[\cos\frac{2(k-l)\pi}{n} + \cos\frac{2(k+l)\pi}{n}\right] \\ &+ \tfrac{1}{2}\left[\cos 2\frac{2(k-l)\pi}{n} + \cos 2\frac{2(k+l)\pi}{n}\right] \\ &+ \cdots\cdots\cdots\cdots \\ &+ \tfrac{1}{2}\left[\cos(n-1)\frac{2(k-l)\pi}{n} + \cos(n-1)\frac{2(k+l)\pi}{n}\right]. \end{aligned}$$

Now the sum of all of the terms in the first column may be equated to $\frac{1}{2}C_{k-l}$ if we again make use of the notation C_k defined by equations (7) and (8). Similarly we observe that the sum of the terms in the second column is $\frac{1}{2}C_{k+l}$. Therefore

$$C_{kl} = \tfrac{1}{2}(C_{k-l}+C_{k+l}). \tag{10}$$

Consider now the expression, analogous to C_{kl},

$$S_{kl} = \sin\frac{2k\pi}{n}\sin\frac{2l\pi}{n} + \sin 2\frac{2k\pi}{n}\sin 2\frac{2l\pi}{n} + \cdots + \sin(n-1)\frac{2k\pi}{n}\sin(n-1)\frac{2l\pi}{n}. \tag{11}$$

Since we have (see Art. 82)

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)],$$

we find

$$\begin{aligned} S_{kl} &= \tfrac{1}{2}\left[\cos\frac{2(k-l)\pi}{n} - \cos\frac{2(k+l)\pi}{n}\right] \\ &+ \tfrac{1}{2}\left[\cos 2\frac{2(k-l)\pi}{n} - \cos 2\frac{2(k+l)\pi}{n}\right] \\ &+ \cdots\cdots\cdots\cdots \\ &+ \tfrac{1}{2}\left[\cos(n-1)\frac{2(k-l)\pi}{n} - \cos(n-1)\frac{2(k+l)\pi}{n},\right. \end{aligned}$$

so that

$$(12) \qquad S_{kl} = \tfrac{1}{2}(C_{k-l} - C_{k+l}).$$

Finally let us put

$$(13) \quad (S_k, C_l) = \sin\frac{2\,k\pi}{n}\cos\frac{2\,l\pi}{n} + \sin 2\,\frac{2\,k\pi}{n}\cos 2\,\frac{2\,l\pi}{n} + \cdots$$
$$+ \sin(n-1)\frac{2\,k\pi}{n}\cos(n-1)\frac{2\,l\pi}{n}.$$

Since we have (see Art. 82)

$$\sin\alpha\cos\beta = \tfrac{1}{2}\left[\sin(\alpha-\beta) + \sin(\alpha+\beta)\right],$$

we find by a repetition of the above method

$$(14) \qquad (S_k, C_l) = \tfrac{1}{2}(S_{k-l} + S_{k+l}).$$

We have already shown (cf. equations (6), (7), and (8)) that

$$(15) \quad \begin{cases} S_k = 0 \text{ for all values of } k, \\ C_k = 0 \text{ for all values of } k \text{ which are } \textit{not} \text{ divisible by } n, \\ C_k = n \text{ for all values of } k \text{ which } \textit{are} \text{ divisible by } n. \end{cases}$$

If we make use of these facts, equations (10), (12), and (14) now teach us that the following statements are true.

$$(16) \quad \begin{cases} C_{kl} = 0 \text{ if } \textit{neither } k-l \textit{ nor } k+l \text{ is divisible by } n, \\ C_{kl} = \dfrac{n}{2} \text{ if } \textit{either } k-l \textit{ or } k+l \text{ is divisible by } n, \text{ but } \textit{not both}, \\ C_{kl} = n \text{ if } \textit{both } k-l \textit{ and } k+l \text{ are divisible by } n. \end{cases}$$

$$(17) \quad \begin{cases} S_{kl} = 0 \text{ if } \textit{neither } k-l \textit{ nor } k+l \text{ is divisible by } n, \\ S_{kl} = +\dfrac{n}{2} \text{ if } k-l \textit{ is} \text{ divisible by } n \text{ and } k+l \textit{ is not} \text{ divisible by } n, \\ S_{kl} = -\dfrac{n}{2} \text{ if } k-l \textit{ is not} \text{ divisible by } n \text{ and } k+l \textit{ is} \text{ divisible by } n, \\ S_{kl} = 0 \text{ if } k-l \text{ and } k+l \text{ are } \textit{both} \text{ divisible by } n. \\ (S_k, C_l) = 0 \text{ for all values of } k \text{ and } l. \end{cases}$$

112. General solution of the problem of trigonometric interpolation. We are now ready for the solution of the problem of trigonometric interpolation. We divide the circumference into $2m+1$ equal parts by means of the angles

$$x = 0,\ \frac{2\pi}{2m+1},\ \frac{4\pi}{2m+1},\ \frac{6\pi}{2m+1},\ \cdots,\ \frac{4m\pi}{2m+1},$$

and denote the corresponding values of the function y by

$$y_0,\ y_1,\ y_2,\ y_3,\ \cdots\ y_{2m}$$

respectively. The function

$$\begin{aligned} y = \tfrac{1}{2} A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx \\ + B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx \end{aligned}$$

will actually assume these values, if the $2m+1$ quantities $A_0, A_1, \cdots, A_m, B_1, B_2, \cdots, B_m$ are chosen so as to satisfy the $2m+1$ equations

$$(1)\quad \begin{cases} y_0 = \tfrac{1}{2} A_0 + A_1 + A_2 + \cdots + A_m, \\ y_1 = \tfrac{1}{2} A_0 + A_1 \cos \dfrac{2\pi}{2m+1} + \cdots + A_m \cos m \dfrac{2\pi}{2m+1} \\ \qquad + B_1 \sin \dfrac{2\pi}{2m+1} + \cdots + B_m \sin m \dfrac{2\pi}{2m+1}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_{2m} = \tfrac{1}{2} A_0 + A_1 \cos \dfrac{4m\pi}{2m+1} + \cdots + A_m \cos m \dfrac{4m\pi}{2m+1} \\ \qquad + B_1 \sin \dfrac{4m\pi}{2m+1} + \cdots + B_m \sin m \dfrac{4m\pi}{2m+1}, \end{cases}$$

all of which may be expressed by the single equation

$$(2)\quad \begin{aligned} y_\nu = \tfrac{1}{2} A_0 + A_1 \cos \frac{2\nu\pi}{2m+1} + \cdots + A_m \cos m \frac{2\nu\pi}{2m+1} \\ + B_1 \sin \frac{2\nu\pi}{2m+1} + \cdots + B_m \sin m \frac{2\nu\pi}{2m+1}, \end{aligned}$$

if we think of ν as assuming in succession the $2m+1$ values

$$\nu = 0, 1, 2, 3, \cdots, 2m.$$

Let r be any integer between 0 and m, let us multiply y_0 by 1, y_1 by $\cos \dfrac{2r\pi}{m+1}$, y_2 by $\cos 2 \dfrac{2r\pi}{m+1}$, $\cdots$, y_ν by

$\cos \nu \dfrac{2 r\pi}{m+1}, \cdots, y_{2m}$ by $\cos 2m \dfrac{2 r\pi}{2m+1}$, and add. If we make use of equations (1) and the notations introduced in Art. 111, we find

$$(3) \quad y_0 + y_1 \cos \frac{2 r\pi}{2m+1} + \cdots + y_{2m} \cos 2m \frac{2 r\pi}{2m+1}$$
$$= \frac{1}{2} A_0 C_r + A_0 C_{1r} + A_2 C_{2r} + \cdots + A_m C_{mr}$$
$$+ B_1(S_1, C_r) + B_2(S_2, C_r) + \cdots + B_m(S_m, C_r).$$

From Art. 111, (17), we know that all of the quantities (S_k, C_l) are equal to zero. Since r was chosen as an integer between 0 and m, r can be divisible by $2m+1$ (which number corresponds to the n of equations (15), (16), (17) of Art. 111), only if $r = 0$. In that case we shall have, according to (15) of Art. 111,

$$C_r = C_0 = n = 2m + 1,$$

and according to (16), Art. 111,

$$C_{10} = C_{20} = \cdots = C_{m0} = 0,$$

since none of the numbers $1, 2, \cdots, m$ are divisible by $2m+1$. Consequently, equation (3) reduces to

$$(4) \qquad y_0 + y_1 + y_2 + \cdots + y_{2m} = \tfrac{1}{2} A_0 (2m + 1)$$

in the case $r = 0$.

If $r > 0$, $C_r = 0$. Moreover, C_{kr} will be zero for all values of k for which neither $k - r$ nor $k + r$ is divisible by $2m+1$. But k as well as r can at most be equal to m, so that neither $k - r$ nor $k + r$ is ever as large as $2m + 1$. The only case therefore in which one of these numbers can be divisible by $2m + 1$ is when $k = r$. Thus we have, according to (16), Art. 111,

$$C_{kr} = 0 \text{ for } k \text{ different from } r,$$
$$C_{rr} = \frac{2m+1}{2}.$$

Consequently equation (3) gives, for $r > 0$,

$$(5)\quad y_0 + y_1 \cos \frac{2r\pi}{2m+1} + y_2 \cos 2\frac{2r\pi}{2m+1} + \cdots + y_{2m} \cos 2m\frac{2r\pi}{2m+1} = \frac{2m+1}{2} A_r,$$

and we notice that equation (4) may be thought of as included in (5) for $r = 0$.* We find therefore the formula

$$(6)\quad A_r = \frac{2}{2m+1}\left[y_0 + y_1 \cos \frac{2r\pi}{2m+1} + y_2 \cos 2\frac{2r\pi}{2m+1} + \cdots + y_{2m} \cos 2m\frac{2r\pi}{2m+1}\right]$$

$$(r = 0, 1, 2, \cdots, m),$$

enabling us to compute $A_0, A_1, \cdots, A_m$ in terms of the given quantities $y_0, y_1, \cdots, y_{2m}$.

It remains to find a corresponding formula for B_r. In order to do this we return to equations (1), multiply them in order by 0, $\sin\frac{2r\pi}{2m+1}$, $\sin 2\frac{2r\pi}{2m+1}, \cdots, \sin 2m\frac{2r\pi}{2m+1}$, and add. This gives

$$y_1 \sin\frac{2r\pi}{2m+1} + y_2 \sin 2\frac{2r\pi}{2m+1} + \cdots + y_{2m} \sin 2m\frac{2r\pi}{2m+1}$$
$$= \tfrac{1}{2}A_0 S_r + A_1(S_r, C_1) + A_2(S_r, C_2) + \cdots + A_m(S_r, C_m)$$
$$+ B_1 S_{1r} + B_2 S_{2r} + \cdots + B_m S_{mr}.$$

All of the terms in the right member of this equation reduce to zero except $B_r S_{rr}$, which, according to Art. 111, (17), becomes equal to $B_r\frac{2m+1}{2}$. Consequently we find

$$(7)\quad B_r = \frac{2}{2m+1}\left[y_1 \sin\frac{2r\pi}{2m+1} + y_2 \sin 2\frac{2r\pi}{2m+1} + \cdots + y_{2m} \sin 2m\frac{2r\pi}{2m+1}\right]$$

$$(r = 1, 2, 3, \cdots, m).$$

* It was for this reason that the notation $\frac{1}{2}A_0$, rather than A_0, was chosen in Art. 108 for the constant term of the harmonic function.

Equations (6) and (7) furnish the complete solution of the problem of trigonometric interpolation. *The coefficients of the harmonic function*

$$y = \tfrac{1}{2}A_0 + A_1 \cos x + A_2 \cos 2x + \cdots + A_m \cos mx$$
$$+ B_1 \sin x + B_2 \sin 2x + \cdots + B_m \sin mx$$

which assumes the arbitrarily assigned values

$$y = y_0, y_1, y_2, \cdots, y_{2m}$$

for the $2m+1$ *equidistant values of the argument*

$$x = 0, \quad \frac{2\pi}{2m+1}, \quad \frac{4\pi}{2m+1}, \ldots\ldots, \frac{4m\pi}{2m+1},$$

are given by equations (6) *and* (7).

EXERCISE LXII

1. Solve the problem treated in detail in Art. 110 by the method of Art. 112.

2. Solve the problems of Exercise LXI by the general method of Art. 112.

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LOGARITHMIC AND TRIGONOMETRIC

TABLES

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TABLE I

FIVE-PLACE LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
101	432	475	518	561	604	647	689	732	775	817
102	860	903	945	988	*030	*072	*115	*157	*199	*242
103	01 284	326	368	410	452	494	536	578	620	662
104	703	745	787	828	870	912	953	995	*036	*078
105	02 119	160	202	243	284	325	366	407	449	490
106	531	572	612	653	694	735	776	816	857	898
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302
108	03 342	383	423	463	503	543	583	623	663	703
109	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
111	532	571	610	650	689	727	766	805	844	883
112	922	961	999	*038	*077	*115	*154	*192	*231	*269
113	05 308	346	385	423	461	500	538	576	614	652
114	690	729	767	805	843	881	918	956	994	*032
115	06 070	108	145	183	221	258	296	333	371	408
116	446	483	521	558	595	633	670	707	744	781
117	819	856	893	930	967	*004	*041	*078	*115	*151
118	07 188	225	262	298	335	372	408	445	482	518
119	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
121	08 279	314	350	386	422	458	493	529	565	600
122	636	672	707	743	778	814	849	884	920	955
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
124	09 342	377	412	447	482	517	552	587	621	656
125	691	726	760	795	830	864	899	934	968	*003
126	10 037	072	106	140	175	209	243	278	312	346
127	380	415	449	483	517	551	585	619	653	687
128	721	755	789	823	857	890	924	958	992	*025
129	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
131	727	760	793	826	860	893	926	959	992	*024
132	12 057	090	123	156	189	222	254	287	320	352
133	385	418	450	483	516	548	581	613	646	678
134	710	743	775	808	840	872	905	937	969	*001
135	13 033	066	098	130	162	194	226	258	290	322
136	354	386	418	450	481	513	545	577	609	640
137	672	704	735	767	799	830	862	893	925	956
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
139	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
141	922	953	983	*014	*045	*076	*106	*137	*168	*198
142	15 229	259	290	320	351	381	412	442	473	503
143	534	564	594	625	655	685	715	746	776	806
144	836	866	897	927	957	987	*017	*047	*077	*107
145	16 137	167	197	227	256	286	316	346	376	406
146	435	465	495	524	554	584	613	643	673	702
147	732	761	791	820	850	879	909	938	967	997
148	17 026	056	085	114	143	173	202	231	260	289
149	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N	0	1	2	3	4	5	6	7	8	9

PP

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	35	34	33
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

PP

N	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
151	898	926	955	984	*013	*041	*070	*099	*127	*156
152	18 184	213	241	270	298	327	355	384	412	441
153	469	498	526	554	583	611	639	667	696	724
154	752	780	808	837	865	893	921	949	977	*005
155	19 033	061	089	117	145	173	201	229	257	285
156	312	340	368	396	424	451	479	507	535	562
157	590	618	645	673	700	728	756	783	811	838
158	866	893	921	948	976	*003	*030	*058	*085	*112
159	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
161	683	710	737	763	790	817	844	871	898	925
162	952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21 219	245	272	299	325	352	378	405	431	458
164	484	511	537	564	590	617	643	669	696	722
165	748	775	801	827	854	880	906	932	958	985
166	22 011	037	063	089	115	141	167	194	220	246
167	272	298	324	350	376	401	427	453	479	505
168	531	557	583	608	634	660	686	712	737	763
169	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
171	300	325	350	376	401	426	452	477	502	528
172	553	578	603	629	654	679	704	729	754	779
173	805	830	855	880	905	930	955	980	*005	*030
174	24 055	080	105	130	155	180	204	229	254	279
175	304	329	353	378	403	428	452	477	502	527
176	551	576	601	625	650	674	699	724	748	773
177	797	822	846	871	895	920	944	969	993	*018
178	25 042	066	091	115	139	164	188	212	237	261
179	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
181	768	792	816	840	864	888	912	935	959	983
182	26 007	031	055	079	102	126	150	174	198	221
183	245	269	293	316	340	364	387	411	435	458
184	482	505	529	553	576	600	623	647	670	694
185	717	741	764	788	811	834	858	881	905	928
186	951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27 184	207	231	254	277	300	323	346	370	393
188	416	439	462	485	508	531	554	577	600	623
189	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
191	28 103	126	149	171	194	217	240	262	285	307
192	330	353	375	398	421	443	466	488	511	533
193	556	578	601	623	646	668	691	713	735	758
194	780	803	825	847	870	892	914	937	959	981
195	29 003	026	048	070	092	115	137	159	181	203
196	226	248	270	292	314	336	358	380	403	425
197	447	469	491	513	535	557	579	601	623	645
198	667	688	710	732	754	776	798	820	842	863
199	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
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P P

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7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

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7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

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8	20.0
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7	16.8	16 1
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7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

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200	30 103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514
202	535	557	578	600	621	643	664	685	707	728
203	750	771	792	814	835	856	878	899	920	942
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154
205	31 175	197	218	239	260	281	302	323	345	366
206	387	408	429	450	471	492	513	534	555	576
207	597	618	639	660	681	702	723	744	765	785
208	806	827	848	869	890	911	931	952	973	994
209	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
211	428	449	469	490	510	531	552	572	593	613
212	634	654	675	695	715	736	756	777	797	818
213	838	858	879	899	919	940	960	980	*001	*021
214	33 041	062	082	102	122	143	163	183	203	224
215	244	264	284	304	325	345	365	385	405	425
216	445	465	486	506	526	546	566	586	606	626
217	646	666	686	706	726	746	766	786	806	826
218	846	866	885	905	925	945	965	985	*005	*025
219	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
221	439	459	479	498	518	537	557	577	596	616
222	635	655	674	694	713	733	753	772	792	811
223	830	850	869	889	908	928	947	967	986	*005
224	35 025	044	064	083	102	122	141	160	180	199
225	218	238	257	276	295	315	334	353	372	392
226	411	430	449	468	488	507	526	545	564	583
227	603	622	641	660	679	698	717	736	755	774
228	793	813	832	851	870	889	908	927	946	965
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
231	361	380	399	418	436	455	474	493	511	530
232	549	568	586	605	624	642	661	680	698	717
233	736	754	773	791	810	829	847	866	884	903
234	922	940	959	977	996	*014	*033	*051	*070	*088
235	37 107	125	144	162	181	199	218	236	254	273
236	291	310	328	346	365	383	401	420	438	457
237	475	493	511	530	548	566	585	603	621	639
238	658	676	694	712	731	749	767	785	803	822
239	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
241	202	220	238	256	274	292	310	328	346	364
242	382	399	417	435	453	471	489	507	525	543
243	561	578	596	614	632	650	668	686	703	721
244	739	757	775	792	810	828	846	863	881	899
245	917	934	952	970	987	*005	*023	*041	*058	*076
246	39 094	111	129	146	164	182	199	217	235	252
247	270	287	305	322	340	358	375	393	410	428
248	445	463	480	498	515	533	550	568	585	602
249	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
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P P

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9	19.8	18.9

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7	13.3
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9	17.1

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7	12.6
8	14.4
9	16.2

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4	6.8
5	8.5
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7	11.9
8	13.6
0	15.3

P P

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250	39 794	811	829	846	863	881	898	915	933	950	
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	
252	40 140	157	175	192	209	226	243	261	278	295	
253	312	329	346	364	381	398	415	432	449	466	
254	483	500	518	535	552	569	586	603	620	637	
255	654	671	688	705	722	739	756	773	790	807	
256	824	841	858	875	892	909	926	943	960	976	
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	
258	41 162	179	196	212	229	246	263	280	296	313	
259	330	347	363	380	397	414	430	447	464	481	
260	497	514	531	547	564	581	597	614	631	647	
261	664	681	697	714	731	747	764	780	797	814	
262	830	847	863	880	896	913	929	946	963	979	
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	
264	42 160	177	193	210	226	243	259	275	292	308	
265	325	341	357	374	390	406	423	439	455	472	
266	488	504	521	537	553	570	586	602	619	635	
267	651	667	684	700	716	732	749	765	781	797	
268	813	830	846	862	878	894	911	927	943	959	
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	
270	43 136	152	169	185	201	217	233	249	265	281	
271	297	313	329	345	361	377	393	409	425	441	
272	457	473	489	505	521	537	553	569	584	600	
273	616	632	648	664	680	696	712	727	743	759	
274	775	791	807	823	838	854	870	886	902	917	
275	933	949	965	981	996	*012	*028	*044	*059	*075	
276	44 091	107	122	138	154	170	185	201	217	232	
277	248	264	279	295	311	326	342	358	373	389	
278	404	420	436	451	467	483	498	514	529	545	
279	560	576	592	607	623	638	654	669	685	700	
280	716	731	747	762	778	793	809	824	840	855	
281	871	886	902	917	932	948	963	979	994	*010	
282	45 025	040	056	071	086	102	117	133	148	163	
283	179	194	209	225	240	255	271	286	301	317	
284	332	347	362	378	393	408	423	439	454	469	
285	484	500	515	530	545	561	576	591	606	621	
286	637	652	667	682	697	712	728	743	758	773	
287	788	803	818	834	849	864	879	894	909	924	
288	939	954	969	984	*000	*015	*030	*045	*060	*075	
289	46 090	105	120	135	150	165	180	195	210	225	
290	240	255	270	285	300	315	330	345	359	374	
291	389	404	419	434	449	464	479	494	509	523	
292	538	553	568	583	598	613	627	642	657	672	
293	687	702	716	731	746	761	776	790	805	820	
294	835	850	864	879	894	909	923	938	953	967	
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	
296	47 129	144	159	173	188	202	217	232	246	261	
297	276	290	305	319	334	349	363	378	392	407	
298	422	436	451	465	480	494	509	524	538	553	
299	567	582	596	611	625	640	654	669	683	698	
300	712	727	741	756	770	784	799	813	828	842	
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PP

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4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

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300	47 712	727	741	756	770	784	799	813	828	842
301	857	871	885	900	914	929	943	958	972	986
302	48 001	015	029	044	058	073	087	101	116	130
303	144	159	173	187	202	216	230	244	259	273
304	287	302	316	330	344	359	373	387	401	416
305	430	444	458	473	487	501	515	530	544	558
306	572	586	601	615	629	643	657	671	686	700
307	714	728	742	756	770	785	799	813	827	841
308	855	869	883	897	911	926	940	954	968	982
309	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49 136	150	164	178	192	206	220	234	248	262
311	276	290	304	318	332	346	360	374	388	402
312	415	429	443	457	471	485	499	513	527	541
313	554	568	582	596	610	624	638	651	665	679
314	693	707	721	734	748	762	776	790	803	817
315	831	845	859	872	886	900	914	927	941	955
316	969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50 106	120	133	147	161	174	188	202	215	229
318	243	256	270	284	297	311	325	338	352	365
319	379	393	406	420	433	447	461	474	488	501
320	515	529	542	556	569	583	596	610	623	637
321	651	664	678	691	705	718	732	745	759	772
322	786	799	813	826	840	853	866	880	893	907
323	920	934	947	961	974	987	*001	*014	*028	*041
324	51 055	068	081	095	108	121	135	148	162	175
325	188	202	215	228	242	255	268	282	295	308
326	322	335	348	362	375	388	402	415	428	441
327	455	468	481	495	508	521	534	548	561	574
328	587	601	614	627	640	654	667	680	693	706
329	720	733	746	759	772	786	799	812	825	838
330	851	865	878	891	904	917	930	943	957	970
331	983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52 114	127	140	153	166	179	192	205	218	231
333	244	257	270	284	297	310	323	336	349	362
334	375	388	401	414	427	440	453	466	479	492
335	504	517	530	543	556	569	582	595	608	621
336	634	647	660	673	686	699	711	724	737	750
337	763	776	789	802	815	827	840	853	866	879
338	892	905	917	930	943	956	969	982	994	*007
339	53 020	033	046	058	071	084	097	110	122	135
340	148	161	173	186	199	212	224	237	250	263
341	275	288	301	314	326	339	352	364	377	390
342	403	415	428	441	453	466	479	491	504	517
343	529	542	555	567	580	593	605	618	631	643
344	656	668	681	694	706	719	732	744	757	769
345	782	794	807	820	832	845	857	870	882	895
346	908	920	933	945	958	970	983	995	*008	*020
347	54 033	045	058	070	083	095	108	120	133	145
348	158	170	183	195	208	220	233	245	258	270
349	283	295	307	320	332	345	357	370	382	394
350	407	419	432	444	456	469	481	494	506	518
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PP

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5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

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350	54 407	419	432	444	456	469	481	494	506	518
351	531	543	555	568	580	593	605	617	630	642
352	654	667	679	691	704	716	728	741	753	765
353	777	790	802	814	827	839	851	864	876	888
354	900	913	925	937	949	962	974	986	998	*011
355	55 023	035	047	060	072	084	096	108	121	133
356	145	157	169	182	194	206	218	230	242	255
357	267	279	291	303	315	328	340	352	364	376
358	388	400	413	425	437	449	461	473	485	497
359	509	522	534	546	558	570	582	594	606	618
360	630	642	654	666	678	691	703	715	727	739
361	751	763	775	787	799	811	823	835	847	859
362	871	883	895	907	919	931	943	955	967	979
363	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
364	56 110	122	134	146	158	170	182	194	205	217
365	229	241	253	265	277	289	301	312	324	336
366	348	360	372	384	396	407	419	431	443	455
367	467	478	490	502	514	526	538	549	561	573
368	585	597	608	620	632	644	656	667	679	691
369	703	714	726	738	750	761	773	785	797	808
370	820	832	844	855	867	879	891	902	914	926
371	937	949	961	972	984	996	*008	*019	*031	*043
372	57 054	066	078	089	101	113	124	136	148	159
373	171	183	194	206	217	229	241	252	264	276
374	287	299	310	322	334	345	357	368	380	392
375	403	415	426	438	449	461	473	484	496	507
376	519	530	542	553	565	576	588	600	611	623
377	634	646	657	669	680	692	703	715	726	738
378	749	761	772	784	795	807	818	830	841	852
379	864	875	887	898	910	921	933	944	955	967
380	978	990	*001	*013	*024	*035	*047	*058	*070	*081
381	58 092	104	115	127	138	149	161	172	184	195
382	206	218	229	240	252	263	274	286	297	309
383	320	331	343	354	365	377	388	399	410	422
384	433	444	456	467	478	490	501	512	524	535
385	546	557	569	580	591	602	614	625	636	647
386	659	670	681	692	704	715	726	737	749	760
387	771	782	794	805	816	827	838	850	861	872
388	883	894	906	917	928	939	950	961	973	984
389	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
390	59 106	118	129	140	151	162	173	184	195	207
391	218	229	240	251	262	273	284	295	306	318
392	329	340	351	362	373	384	395	406	417	428
393	439	450	461	472	483	494	506	517	528	539
394	550	561	572	583	594	605	616	627	638	649
395	660	671	682	693	704	715	726	737	748	759
396	770	780	791	802	813	824	835	846	857	868
397	879	890	901	912	923	934	945	956	966	977
398	988	999	*010	*021	*032	*043	*054	*065	*076	*086
399	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
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P P

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8	9.6
9	10.8

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	10
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P P

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400	60 206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412
402	423	433	444	455	466	477	487	498	509	520
403	531	541	552	563	574	584	595	606	617	627
404	638	649	660	670	681	692	703	713	724	735
405	746	756	767	778	788	799	810	821	831	842
406	853	863	874	885	895	906	917	927	938	949
407	959	970	981	991	*002	*013	*023	*034	*045	*055
408	61 066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
411	384	395	405	416	426	437	448	458	469	479
412	490	500	511	521	532	542	553	563	574	584
413	595	606	616	627	637	648	658	669	679	690
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418	118	128	138	149	159	170	180	190	201	211
419	221	232	242	252	263	273	284	294	304	315
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421	428	439	449	459	469	480	490	500	511	521
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423	634	644	655	665	675	685	696	706	716	726
424	737	747	757	767	778	788	798	808	818	829
425	839	849	859	870	880	890	900	910	921	931
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427	63 043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236
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436	949	959	969	979	988	998	*008	*018	*028	*038
437	64 048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237
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443	640	650	660	670	680	689	699	709	719	729
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547	799	807	815	823	830	838	846	854	862	870
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676	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
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690	885	891	897	904	910	916	923	929	935	942
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697	323	330	336	342	348	354	361	367	373	379
698	386	392	398	404	410	417	423	429	435	442
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704	757	763	770	776	782	788	794	800	807	813
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717	552	558	564	570	576	582	588	594	600	606
718	612	618	625	631	637	643	649	655	661	667
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747	332	338	344	349	355	361	367	373	379	384
748	390	396	402	408	413	419	425	431	437	442
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758	967	973	978	984	990	996	*001	*007	*013	*018
759	88 024	030	036	041	047	053	058	064	070	076
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762	195	201	207	213	218	224	230	235	241	247
763	252	258	264	270	275	281	287	292	298	304
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765	366	372	377	383	389	395	400	406	412	417
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783	376	382	387	393	398	404	409	415	421	426
784	432	437	443	448	454	459	465	470	476	481
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787	597	603	609	614	620	625	631	636	642	647
788	653	658	664	669	675	680	686	691	697	702
789	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
791	818	823	829	834	840	845	851	856	862	867
792	873	878	883	889	894	900	905	911	916	922
793	927	933	938	944	949	955	960	966	971	977
794	982	988	993	998	*004	*009	*015	*020	*026	*031
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798	200	206	211	217	222	227	233	238	244	249
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800	309	314	320	325	331	336	342	347	352	358

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823	540	545	551	556	561	566	572	577	582	587
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825	645	651	656	661	666	672	677	682	687	693
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844	634	639	645	650	655	660	665	670	675	681
845	686	691	696	701	706	711	716	722	727	732
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961	272	277	281	286	290	295	299	304	308	313
962	318	322	327	331	336	340	345	349	354	358
963	363	367	372	376	381	385	390	394	399	403
964	408	412	417	421	426	430	435	439	444	448
965	453	457	462	466	471	475	480	484	489	493
966	498	502	507	511	516	520	525	529	534	538
967	543	547	552	556	561	565	570	574	579	583
968	588	592	597	601	605	610	614	619	623	628
969	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
971	722	726	731	735	740	744	749	753	758	762
972	767	771	776	780	784	789	793	798	802	807
973	811	816	820	825	829	834	838	843	847	851
974	856	860	865	869	874	878	883	887	892	896
975	900	905	909	914	918	923	927	932	936	941
976	945	949	954	958	963	967	972	976	981	985
977	989	994	998	*003	*007	*012	*016	*021	*025	*029
978	99 034	038	043	047	052	056	061	065	069	074
979	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
981	167	171	176	180	185	189	193	198	202	207
982	211	216	220	224	229	233	238	242	247	251
983	255	260	264	269	273	277	282	286	291	295
984	300	304	308	313	317	322	326	330	335	339
985	344	348	352	357	361	366	370	374	379	383
986	388	392	396	401	405	410	414	419	423	427
987	432	436	441	445	449	454	458	463	467	471
988	476	480	484	489	493	498	502	506	511	515
989	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
991	607	612	616	621	625	629	634	638	642	647
992	651	656	660	664	669	673	677	682	686	691
993	695	699	704	708	712	717	721	726	730	734
994	739	743	747	752	756	760	765	769	774	778
995	782	787	791	795	800	804	808	813	817	822
996	826	830	835	839	843	848	852	856	861	865
997	870	874	878	883	887	891	896	900	904	909
998	913	917	922	926	930	935	939	944	948	952
999	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039
N	0	1	2	3	4	5	6	7	8	9

P P

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

N	0	1	2	3	4	5	6	7	8	9
1000	000 0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001	4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002	8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001 3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004	7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002 1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006	5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003 0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008	4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009	8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004 3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011	7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005 1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013	6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006 0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015	4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016	8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007 3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018	7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008 1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020	6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009 0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022	4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023	8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010 3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025	7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011 1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027	5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028	9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012 4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030	8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013 2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032	6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014 1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034	5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035	9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015 3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037	7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016 1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039	6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017 0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041	4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042	8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018 2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044	7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019 1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1045	5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047	9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020 3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049	7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
1050	021 1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051	6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022 0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053	4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054	8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023 2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056	6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024 0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058	4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059	8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025 3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061	7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026 1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063	5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064	9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027 3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066	7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028 1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068	5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069	9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029 3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071	7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030 1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073	5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031 0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075	4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076	8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032 2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078	6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033 0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080	4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081	8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034 2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083	6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1064	035 0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085	4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086	8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036 2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088	6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037 0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090	4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091	8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038 2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093	6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039 0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095	4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096	8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040 2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098	6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099	9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041 3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N	0	1	2	3	4	5	6	7	8	9

TABLE II

FIVE-PLACE LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

FOR EVERY MINUTE OF ARC FROM 0° TO 90°

0°*

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	——		——		——	0.00 000	60
1	6.46 373	30103	6.46 373	30103	3.53 627	0.00 000	59
2	6.76 476	17609	6.76 476	17609	3.23 524	0.00 000	58
3	6.94 085	12494	6.94 085	12494	3.05 915	0.00 000	57
4	7.06 579	9691	7.06 579	9691	2.93 421	0.00 000	56
5	7.16 270	7918	7.16 270	7918	2.83 730	0.00 000	55
6	7.24 188	6694	7.24 188	6694	2.75 812	0.00 000	54
7	7.30 882	5800	7.30 882	5800	2.69 118	0.00 000	53
8	7.36 682	5115	7.36 682	5115	2.63 318	0.00 000	52
9	7.41 797	4576	7.41 797	4576	2.58 203	0.00 000	51
10	7.46 373	4139	7.46 373	4139	2.53 627	0.00 000	50
11	7.50 512	3779	7.50 512	3779	2.49 488	0.00 000	49
12	7.54 291	3476	7.54 291	3476	2.45 709	0.00 000	48
13	7.57 767	3218	7.57 767	3219	2.42 233	0.00 000	47
14	7.60 985	2997	7.60 986	2996	2.39 014	0.00 000	46
15	7.63 982	2802	7.63 982	2803	2.36 018	0.00 000	45
16	7.66 784	2633	7.66 785	2633	2.33 215	0.00 000	44
17	7.69 417	2483	7.69 418	2482	2.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	2.28 100	9.99 999	42
19	7.74 248	2227	7.74 248	2228	2.25 752	9.99 999	41
20	7.76 475	2119	7.76 476	2119	2.23 524	9.99 999	40
21	7.78 594	2021	7.78 595	2020	2.21 405	9.99 999	39
22	7.80 615	1930	7.80 615	1931	2.19 385	9.99 999	38
23	7.82 545	1848	7.82 546	1848	2.17 454	9.99 999	37
24	7.84 393	1773	7.84 394	1773	2.15 606	9.99 999	36
25	7.86 166	1704	7.86 167	1704	2.13 833	9.99 999	35
26	7.87 870	1639	7.87 871	1639	2.12 129	9.99 999	34
27	7.89 509	1579	7.89 510	1579	2.10 490	9.99 999	33
28	7.91 088	1524	7.91 089	1524	2.08 911	9.99 999	32
29	7.92 612	1472	7.92 613	1473	2.07 387	9.99 998	31
30	7.94 084	1424	7.94 086	1424	2.05 914	9.99 998	30
31	7.95 508	1379	7.95 510	1379	2.04 490	9.99 998	29
32	7.96 887	1336	7.96 889	1336	2.03 111	9.99 998	28
33	7.98 223	1297	7.98 225	1297	2.01 775	9.99 998	27
34	7.99 520	1259	7.99 522	1259	2.00 478	9.99 998	26
35	8.00 779	1223	8.00 781	1223	1.99 219	9.99 998	25
36	8.02 002	1190	8.02 004	1190	1.97 996	9.99 998	24
37	8.03 192	1158	8.03 194	1159	1.96 806	9.99 997	23
38	8.04 350	1128	8.04 353	1128	1.95 647	9.99 997	22
39	8.05 478	1100	8.05 481	1100	1.94 519	9.99 997	21
40	8.06 578	1072	8.06 581	1072	1.93 419	9.99 997	20
41	8.07 650	1046	8.07 653	1047	1.92 347	9.99 997	19
42	8.08 696	1022	8.08 700	1022	1.91 300	9.99 997	18
43	8.09 718	999	8.09 722	998	1.90 278	9.99 997	17
44	8.10 717	976	8.10 720	976	1.89 280	9.99 996	16
45	8.11 693	954	8.11 696	955	1.88 304	9.99 996	15
46	8.12 647	934	8.12 651	934	1.87 349	9.99 996	14
47	8.13 581	914	8.13 585	915	1.86 415	9.99 996	13
48	8.14 495	896	8.14 500	895	1.85 500	9.99 996	12
49	8.15 391	877	8.15 395	878	1.84 605	9.99 996	11
50	8.16 268	860	8.16 273	860	1.83 727	9.99 995	10
51	8.17 128	843	8.17 133	843	1.82 867	9.99 995	9
52	8.17 971	827	8.17 976	828	1.82 024	9.99 995	8
53	8.18 798	812	8.18 804	812	1.81 196	9.99 995	7
54	8.19 610	797	8.19 616	797	1.80 384	9.99 995	6
55	8.20 407	782	8.20 413	782	1.79 587	9.99 994	5
56	8.21 189	769	8.21 195	769	1.78 805	9.99 994	4
57	8.21 958	755	8.21 964	756	1.78 036	9.99 994	3
58	8.22 713	743	8.22 720	742	1.77 280	9.99 994	2
59	8.23 456	730	8.23 462	730	1.76 538	9.99 994	1
60	8.24 186		8.24 192		1.75 808	9.99 993	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

89°*

* If interpolation is necessary, use table III or IV.

1°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.24 186		8.24 192		1.75 808	9.99 993	60
1	8.24 903	717	8.24 910	718	1.75 090	9.99 993	59
2	8.25 609	706	8.25 616	706	1.74 384	9.99 993	58
3	8.26 304	695	8.26 312	696	1.73 688	9.99 993	57
4	8.26 988	684	8.26 996	684	1.73 004	9.99 992	56
5	8.27 661	673	8.27 669	673	1.72 331	9.99 992	55
6	8.28 324	663	8.28 332	663	1.71 668	9.99 992	54
7	8.28 977	653	8.28 986	654	1.71 014	9.99 992	53
8	8.29 621	644	8.29 629	643	1.70 371	9.99 992	52
9	8.30 255	634	8.30 263	634	1.69 737	9.99 991	51
10	8.30 879	624	8.30 888	625	1.69 112	9.99 991	50
11	8.31 495	616	8.31 505	617	1.68 495	9.99 991	49
12	8.32 103	608	8.32 112	607	1.67 888	9.99 990	48
13	8.32 702	599	8.32 711	599	1.67 289	9.99 990	47
14	8.33 292	590	8.33 302	591	1.66 698	9.99 990	46
15	8.33 875	583	8.33 886	584	1.66 114	9.99 990	45
16	8.34 450	575	8.34 461	575	1.65 539	9.99 989	44
17	8.35 018	568	8.35 029	568	1.64 971	9.99 989	43
18	8.35 578	560	8.35 590	561	1.64 410	9.99 989	42
19	8.36 131	553	8.36 143	553	1.63 857	9.99 989	41
20	8.36 678	547	8.36 689	546	1.63 311	9.99 988	40
21	8.37 217	539	8.37 229	540	1.62 771	9.99 988	39
22	8.37 750	533	8.37 762	533	1.62 238	9.99 988	38
23	8.38 276	526	8.38 289	527	1.61 711	9.99 987	37
24	8.38 796	520	8.38 809	520	1.61 191	9.99 987	36
25	8.39 310	514	8.39 323	514	1.60 677	9.99 987	35
26	8.39 818	508	8.39 832	509	1.60 168	9.99 986	34
27	8.40 320	502	8.40 334	502	1.59 666	9.99 986	33
28	8.40 816	496	8.40 830	496	1.59 170	9.99 986	32
29	8.41 307	491	8.41 321	491	1.58 679	9.99 985	31
30	8.41 792	485	8.41 807	486	1.58 193	9.99 985	30
31	8.42 272	480	8.42 287	480	1.57 713	9.99 985	29
32	8.42 746	474	8.42 762	475	1.57 238	9.99 984	28
33	8.43 216	470	8.43 232	470	1.56 768	9.99 984	27
34	8.43 680	464	8.43 696	464	1.56 304	9.99 984	26
35	8.44 139	459	8.44 156	460	1.55 844	9.99 983	25
36	8.44 594	455	8.44 611	455	1.55 389	9.99 983	24
37	8.45 044	450	8.45 061	450	1.54 939	9.99 983	23
38	8.45 489	445	8.45 507	446	1.54 493	9.99 982	22
39	8.45 930	441	8.45 948	441	1.54 052	9.99 982	21
40	8.46 366	436	8.46 385	437	1.53 615	9.99 982	20
41	8.46 799	433	8.46 817	432	1.53 183	9.99 981	19
42	8.47 226	427	8.47 245	428	1.52 755	9.99 981	18
43	8.47 650	424	8.47 669	424	1.52 331	9.99 981	17
44	8.48 069	419	8.48 089	420	1.51 911	9.99 980	16
45	8.48 485	416	8.48 505	416	1.51 495	9.99 980	15
46	8.48 896	411	8.48 917	412	1.51 083	9.99 979	14
47	8.49 304	408	8.49 325	408	1.50 675	9.99 979	13
48	8.49 708	404	8.49 729	404	1.50 271	9.99 979	12
49	8.50 108	400	8.50 130	401	1.49 870	9.99 978	11
50	8.50 504	396	8.50 527	397	1.49 473	9.99 978	10
51	8.50 897	393	8.50 920	393	1.49 080	9.99 977	9
52	8.51 287	390	8.51 310	390	1.48 690	9.99 977	8
53	8.51 673	386	8.51 696	386	1.48 304	9.99 977	7
54	8.52 055	382	8.52 079	383	1.47 921	9.99 976	6
55	8.52 434	379	8.52 459	380	1.47 541	9.99 976	5
56	8.52 810	376	8.52 835	376	1.47 165	9.99 975	4
57	8.53 183	373	8.53 208	373	1.46 792	9.99 975	3
58	8.53 552	369	8.53 578	370	1.46 422	9.99 974	2
59	8.53 919	367	8.53 945	367	1.46 055	9.99 974	1
60	8.54 282	363	8.54 308	363	1.45 692	9.99 974	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

88° *

* If interpolation is necessary, use table III or IV.

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.54 282		8.54 308		1.45 692	9.99 974	60
1	8.54 642	360	8.54 669	361	1.45 331	9.99 973	59
2	8.54 999	357	8.55 027	358	1.44 973	9.99 973	58
3	8.55 354	355	8.55 382	355	1.44 618	9.99 972	57
4	8.55 705	351	8.55 734	352	1.44 266	9.99 972	56
5	8.56 054	349	8.56 083	349	1.43 917	9.99 971	55
6	8.56 400	346	8.56 429	346	1.43 571	9.99 971	54
7	8.56 743	343	8.56 773	344	1.43 227	9.99 970	53
8	8.57 084	341	8.57 114	341	1.42 886	9.99 970	52
9	8.57 421	337	8.57 452	338	1.42 548	9.99 969	51
10	8.57 757	336	8.57 788	336	1.42 212	9.99 969	50
11	8.58 089	332	8.58 121	333	1.41 879	9.99 968	49
12	8.58 419	330	8.58 451	330	1.41 549	9.99 968	48
13	8.58 747	328	8.58 779	328	1.41 221	9.99 967	47
14	8.59 072	325	8.59 105	326	1.40 895	9.99 967	46
15	8.59 395	323	8.59 428	323	1.40 572	9.99 967	45
16	8.59 715	320	8.59 749	321	1.40 251	9.99 966	44
17	8.60 033	318	8.60 068	319	1.39 932	9.99 966	43
18	8.60 349	316	8.60 384	316	1.39 616	9.99 965	42
19	8.60 662	313	8.60 698	314	1.39 302	9.99 964	41
20	8.60 973	311	8.61 009	311	1.38 991	9.99 964	40
21	8.61 282	309	8.61 319	310	1.38 681	9.99 963	39
22	8.61 589	307	8.61 626	307	1.38 374	9.99 963	38
23	8.61 894	305	8.61 931	305	1.38 069	9.99 962	37
24	8.62 196	302	8.62 234	303	1.37 766	9.99 962	36
25	8.62 497	301	8.62 535	301	1.37 465	9.99 961	35
26	8.62 795	298	8.62 834	299	1.37 166	9.99 961	34
27	8.63 091	296	8.63 131	297	1.36 869	9.99 960	33
28	8.63 385	294	8.63 426	295	1.36 574	9.99 960	32
29	8.63 678	293	8.63 718	292	1.36 282	9.99 959	31
30	8.63 968	290	8.64 009	291	1.35 991	9.99 959	30
31	8.64 256	288	8.64 298	289	1.35 702	9.99 958	29
32	8.64 543	287	8.64 585	287	1.35 415	9.99 958	28
33	8.64 827	284	8.64 870	285	1.35 130	9.99 957	27
34	8.65 110	283	8.65 154	284	1.34 846	9.99 956	26
35	8.65 391	281	8.65 435	281	1.34 565	9.99 956	25
36	8.65 670	279	8.65 715	280	1.34 285	9.99 955	24
37	8.65 947	277	8.65 993	278	1.34 007	9.99 955	23
38	8.66 223	276	8.66 269	276	1.33 731	9.99 954	22
39	8.66 497	274	8.66 543	274	1.33 457	9.99 954	21
40	8.66 769	272	8.66 816	273	1.33 184	9.99 953	20
41	8.67 039	270	8.67 087	271	1.32 913	9.99 952	19
42	8.67 308	269	8.67 356	269	1.32 644	9.99 952	18
43	8.67 575	267	8.67 624	268	1.32 376	9.99 951	17
44	8.67 841	266	8.67 890	266	1.32 110	9.99 951	16
45	8.68 104	263	8.68 154	264	1.31 846	9.99 950	15
46	8.68 367	263	8.68 417	263	1.31 583	9.99 949	14
47	8.68 627	260	8.68 678	261	1.31 322	9.99 949	13
48	8.68 886	259	8.68 938	260	1.31 062	9.99 948	12
49	8.69 144	258	8.69 196	258	1.30 804	9.99 948	11
50	8.69 400	256	8.69 453	257	1.30 547	9.99 947	10
51	8.69 654	254	8.69 708	255	1.30 292	9.99 946	9
52	8.69 907	253	8.69 962	254	1.30 038	9.99 946	8
53	8.70 159	252	8.70 214	252	1.29 786	9.99 945	7
54	8.70 409	250	8.70 465	251	1.29 535	9.99 944	6
55	8.70 658	249	8.70 714	249	1.29 286	9.99 944	5
56	8.70 905	247	8.70 962	248	1.29 038	9.99 943	4
57	8.71 151	246	8.71 208	246	1.28 792	9.99 942	3
58	8.71 395	244	8.71 453	245	1.28 547	9.99 942	2
59	8.71 638	243	8.71 697	244	1.28 303	9.99 941	1
60	8.71 880	242	8.71 940	243	1.28 060	9.99 940	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

P P

	360	350	340
1	36	35	34
2	72	70	68
3	108	105	102
4	144	140	136
5	180	175	170
6	216	210	204
7	252	245	238
8	288	280	272
9	324	315	306

	330	320	310
1	33	32	31
2	66	64	62
3	99	96	93
4	132	128	124
5	165	160	155
6	198	192	186
7	231	224	217
8	264	256	248
9	297	288	279

	300	290	285
1	30	29	28.5
2	60	58	57.0
3	90	87	85.5
4	120	116	114.0
5	150	145	142.5
6	180	174	171.0
7	210	203	199.5
8	240	232	228.0
9	270	261	256.5

	280	275	270
1	28.0	27.5	27.0
2	56.0	55.0	54.0
3	84.0	82.5	81.0
4	112.0	110.0	108.0
5	140.0	137.5	135.0
6	168.0	165.0	162.0
7	196.0	192.5	189.0
8	224.0	220.0	216.0
9	252.0	247.5	243.0

	265	260	255
1	26.5	26.0	25.5
2	53.0	52.0	51.0
3	79.5	78.0	76.5
4	106.0	104.0	102.0
5	132.5	130.0	127.5
6	159.0	156.0	153.0
7	185.5	182.0	178.5
8	212.0	208.0	204.0
9	238.5	234.0	229.5

	250	245	240
1	25.0	24.5	24.0
2	50.0	49.0	48.0
3	75.0	73.5	72.0
4	100.0	98.0	96.0
5	125.0	122.5	120.0
6	150.0	147.0	144.0
7	175.0	171.5	168.0
8	200.0	196.0	192.0
9	225.0	220.5	216.0

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8 72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8 72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	55
6	8.73 303	232	8 73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8 74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	50
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	1.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8 76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8 76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	1.23 475	9.99 926	40
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925	38
23	8 77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8 78 022	210	1.21 978	9.99 921	33
28	8 78 152	208	8.78 232	209	1 21 768	9.99 920	32
29	8,78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8 78 568	206	8.78 649	206	1.21 351	9.99 919	30
31	8 78 774	205	8.78 855	206	1.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	1.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	20
41	8 80 782	196	8 80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1 18 736	9.99 909	17
44	8.81 367	193	8 81 459	194	1.18 541	9 99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8 82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	10
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	1 16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5
56	8.83 630	183	8 83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895	1
60	8 84 358		8.84 464		1.15 536	9.99 894	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

P P

	241	239	237	236	234
1	24.1	23.9	23.7	23.6	23.4
2	48.2	47.8	47.4	47.2	46.8
3	72.3	71.7	71.1	70.8	70.2
4	96.4	95.6	94.8	94.4	93.6
5	120.5	119.5	118.5	118.0	117.0
6	144.6	143.4	142.2	141.6	140.4
7	168.7	167.3	165.9	165.2	163.8
8	192.8	191.2	189.6	188.8	187.2
9	216.9	215.1	213.3	212.4	210.6

	232	231	229	227	226
1	23.2	23.1	22.9	22.7	22.6
2	46.4	46.2	45.8	45.4	45.2
3	69.6	69.3	68.7	68.1	67.8
4	92.8	92.4	91.6	90.8	90.4
5	116.0	115.5	114.5	113.5	113.0
6	139.2	138.6	137.4	136.2	135.6
7	162.4	161.7	160.3	158.9	158.2
8	185.6	184.8	183.2	181.6	180.8
9	208.8	207.9	206.1	204.3	203.4

	224	222	220	219	217
1	22.4	22.2	22.0	21.9	21.7
2	44.8	44.4	44.0	43.8	43.4
3	67.2	66.6	66.0	65.7	65.1
4	89.6	88.8	88.0	87.6	86.8
5	112.0	111.0	110.0	109.5	108.5
6	134.4	133.2	132.0	131.4	130.2
7	156.8	155.4	154.0	153.3	151.9
8	179.2	177.6	176.0	175.2	173.6
9	201.6	199.8	198.0	197.1	195.3

	216	214	213	211	209
1	21.6	21.4	21.3	21.1	20.9
2	43.2	42.8	42.6	42.2	41.8
3	64.8	64.2	63.9	63.3	62.7
4	86.4	85.6	85.2	84.4	83.6
5	108.0	107.0	106.5	105.5	104.5
6	129.6	128.4	127.8	126.6	125.4
7	151.2	149.8	149.1	147.7	146.3
8	172.8	171.2	170.4	168.8	167.2
9	194.4	192.6	191.7	189.9	188.1

	208	206	203	201	199
1	20.8	20.6	20.3	20.1	19.9
2	41.6	41.2	40.6	40.2	39.8
3	62.4	61.8	60.9	60.3	59.7
4	83.2	82.4	81.2	80.4	79.6
5	104.0	103.0	101.5	100.5	99.5
6	124.8	123.6	121.8	120.6	119.4
7	145.6	144.2	142.1	140.7	139.3
8	166.4	164.8	162.4	160.8	159.2
9	187 2	185.4	182.7	180.9	179.1

	198	196	194	192	190
1	19.8	19.6	19.4	19.2	19.0
2	39.6	39.2	38.8	38.4	38.0
3	59.4	58.8	58.2	57.6	57.0
4	79 2	78.4	77.6	76.8	76.0
5	99.0	98.0	97.0	96.0	95.0
6	118.8	117.6	116.4	115.2	114.0
7	138.6	137.2	135.8	134.4	133.0
8	158.4	156.8	155.2	153.6	152.0
9	178 2	176.4	174.6	172.8	171.0

	188	186	184	182	181
1	18.8	18.6	18.4	18.2	18.1
2	37.6	37.2	36.8	36.4	36.2
3	56.4	55.8	55.2	54.6	54.3
4	75.2	74.4	73.6	72.8	72.4
5	94.0	93.0	92.0	91.0	90.5
6	112.8	111.6	110.4	109.2	108.6
7	131.6	130.2	128.8	127.4	126.7
8	150.4	148.8	147.2	145.6	144.8
9	169.2	167.4	165.6	163.8	162.9

P P

4°

′	L Sin	d	L Tan	cd	L Cot	L Cos	
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 185	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 605	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 955	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 065	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 785	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 995	166	8.88 120	167	1.11 880	9.99 875	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
25	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 625	159	8.89 760	160	1.10 240	9.99 865	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 715	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 185	155	1.08 815	9.99 856	20
41	8.91 195	154	8.91 340	155	1.08 660	9.99 855	19
42	8.91 349	153	8.91 495	155	1.08 505	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 565	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 165	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 885	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805	9.99 834	0
	L Cos	d	L Cot	cd	L Tan	L Sin	′

P P

	182	181	180	179	178
1	18.2	18.1	18.0	17.9	17.8
2	36.4	36.2	36.0	35.8	35.6
3	54.6	54.3	54.0	53.7	53.4
4	72.8	72.4	72.0	71.6	71.2
5	91.0	90.5	90.0	89.5	89.0
6	109.2	108.6	108.0	107.4	106.8
7	127.4	126.7	126.0	125.3	124.6
8	145.6	144.8	144.0	143.2	142.4
9	163.8	162.9	162.0	161.1	160.2
	177	**176**	**175**	**174**	**173**
1	17.7	17.6	17.5	17.4	17.3
2	35.4	35.2	35.0	34.8	34.6
3	53.1	52.8	52.5	52.2	51.9
4	70.8	70.4	70.0	69.6	69.2
5	88.5	88.0	87.5	87.0	86.5
6	106.2	105.6	105.0	104.4	103.8
7	123.9	123.2	122.5	121.8	121.1
8	141.6	140.8	140.0	139.2	138.4
9	159.3	158.4	157.5	156.6	155.7
	172	**171**	**170**	**169**	**168**
1	17.2	17.1	17.0	16.9	16.8
2	34.4	34.2	34.0	33.8	33.6
3	51.6	51.3	51.0	50.7	50.4
4	68.8	68.4	68.0	67.6	67.2
5	86.0	85.5	85.0	84.5	84.0
6	103.2	102.6	102.0	101.4	100.8
7	120.4	119.7	119.0	118.3	117.6
8	137.6	136.8	136.0	135.2	134.4
9	154.8	153.9	153.0	152.1	151.2
	167	**166**	**165**	**164**	**163**
1	16.7	16.6	16.5	16.4	16.3
2	33.4	33.2	33.0	32.8	32.6
3	50.1	49.8	49.5	49.2	48.9
4	66.8	66.4	66.0	65.6	65.2
5	83.5	83.0	82.5	82.0	81.5
6	100.2	99.6	99.0	98.4	97.8
7	116.9	116.2	115.5	114.8	114.1
8	133.6	132.8	132.0	131.2	130.4
9	150.3	149.4	148.5	147.6	146.7
	162	**161**	**160**	**159**	**158**
1	16.2	16.1	16.0	15.9	15.8
2	32.4	32.2	32.0	31.8	31.6
3	48.6	48.3	48.0	47.7	47.4
4	64.8	64.4	64.0	63.6	63.2
5	81.0	80.5	80.0	79.5	79.0
6	97.2	96.6	96.0	95.4	94.8
7	113.4	112.7	112.0	111.3	110.6
8	129.6	128.8	128.0	127.2	126.4
9	145.8	144.9	144.0	143.1	142.2
	157	**156**	**155**	**154**	**153**
1	15.7	15.6	15.5	15.4	15.3
2	31.4	31.2	31.0	30.8	30.6
3	47.1	46.8	46.5	46.2	45.9
4	62.8	62.4	62.0	61.6	61.2
5	78.5	78.0	77.5	77.0	76.5
6	94.2	93.6	93.0	92.4	91.8
7	109.9	109.2	108.5	107.8	107.1
8	125.6	124.8	124.0	123.2	122.4
0	141.3	140.4	139.5	138.6	137.7
	152	**151**	**150**	**149**	**148**
1	15.2	15.1	15.0	14.9	14.8
2	30.4	30.2	30.0	29.8	29.6
3	45.6	45.3	45.0	44.7	44.4
4	60.8	60.4	60.0	59.6	59.2
5	76.0	75.5	75.0	74.5	74.0
6	91.2	90.6	90.0	89.4	88.8
7	106.4	105.7	105.0	104.3	103.6
8	121.6	120.8	120.0	119.2	118.4
9	136.8	135.9	135.0	134.1	133.2

P P

85°

′	L Sin	d	L Tan	cd	L Cot	L Cos	
0	8.94 030	144	8.94 195	145	1.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	1.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	1.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
	L Cos	d	L Cot	cd	L Tan	L Sin	′

P P

	147	146	145	144
1	14.7	14.6	14.5	14.4
2	29.4	29.2	29.0	28.8
3	44.1	43.8	43.5	43.2
4	58.8	58.4	58.0	57.6
5	73.5	73.0	72.5	72.0
6	88.2	87.6	87.0	86.4
7	102.9	102.2	101.5	100.8
8	117.6	116.8	116.0	115.2
9	132.3	131.4	130.5	129.6

	143	142	141	140
1	14.3	14.2	14.1	14.0
2	28.6	28.4	28.2	28.0
3	42.9	42.6	42.3	42.0
4	57.2	56.8	56.4	56.0
5	71.5	71.0	70.5	70.0
6	85.8	85.2	84.6	84.0
7	100.1	99.4	98.7	98.0
8	114.4	113.6	112.8	112.0
9	128.7	127.8	126.9	126.0

	139	138	137	136
1	13.9	13.8	13.7	13.6
2	27.8	27.6	27.4	27.2
3	41.7	41.4	41.1	40.8
4	55.6	55.2	54.8	54.4
5	69.5	69.0	68.5	68.0
6	83.4	82.8	82.2	81.6
7	97.3	96.6	95.9	95.2
8	111.2	110.4	109.6	108.8
9	125.1	124.2	123.3	122.4

	135	134	133	132
1	13.5	13.4	13.3	13.2
2	27.0	26.8	26.6	26.4
3	40.5	40.2	39.9	39.6
4	54.0	53.6	53.2	52.8
5	67.5	67.0	66.5	66.0
6	81.0	80.4	79.8	79.2
7	94.5	93.8	93.1	92.4
8	108.0	107.2	106.4	105.6
9	121.5	120.6	119.7	118.8

	131	130	129	128
1	13.1	13.0	12.9	12.8
2	26.2	26.0	25.8	25.6
3	39.3	39.0	38.7	38.4
4	52.4	52.0	51.6	51.2
5	65.5	65.0	64.5	64.0
6	78.6	78.0	77.4	76.8
7	91.7	91.0	90.3	89.6
8	104.8	104.0	103.2	102.4
9	117.9	117.0	116.1	115.2

	127	126	125	124
1	12.7	12.6	12.5	12.4
2	25.4	25.2	25.0	24.8
3	38.1	37.8	37.5	37.2
4	50.8	50.4	50.0	49.6
5	63.5	63.0	62.5	62.0
6	76.2	75.6	75.0	74.4
7	88.9	88.2	87.5	86.8
8	101.6	100.8	100.0	99.2
9	114.3	113.4	112.5	111.6

	123	122	121	120
1	12.3	12.2	12.1	12.0
2	24.6	24.4	24.2	24.0
3	36.9	36.6	36.3	36.0
4	49.2	48.8	48.4	48.0
5	61.5	61.0	60.5	60.0
6	73.8	73.2	72.6	72.0
7	86.1	85.4	84.7	84.0
8	98.4	97.6	96.8	96.0
9	110.7	109.8	108.9	108.0

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755	55
6	9.02 639	118	9.02 885	120	0.97 115	9.99 753	54
7	9.02 757	117	9.03 005	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	45
16	9.03 805	115	9.04 065	116	0.95 935	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275	111	9.05 553	113	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	0.93 225	9.99 705	20
41	9.06 589	107	9.06 885	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

P P

	121	120	119	118
1	12.1	12.0	11.9	11.8
2	24.2	24.0	23.8	23.6
3	36.3	36.0	35.7	35.4
4	48.4	48.0	47.6	47.2
5	60.5	60.0	59.5	59.0
6	72.6	72.0	71.4	70.8
7	84.7	84.0	83.3	82.6
8	96.8	96.0	95.2	94.4
9	108.9	108.0	107.1	106.2

	117	116	115	114
1	11.7	11.6	11.5	11.4
2	23.4	23.2	23.0	22.8
3	35.1	34.8	34.5	34.2
4	46.8	46.4	46.0	45.6
5	58.5	58.0	57.5	57.0
6	70.2	69.6	69.0	68.4
7	81.9	81.2	80.5	79.8
8	93.6	92.8	92.0	91.2
9	105.3	104.4	103.5	102.6

	113	112	111	110
1	11.3	11.2	11.1	11.0
2	22.6	22.4	22.2	22.0
3	33.9	33.6	33.3	33.0
4	45.2	44.8	44.4	44.0
5	56.5	56.0	55.5	55.0
6	67.8	67.2	66.6	66.0
7	79.1	78.4	77.7	77.0
8	90.4	89.6	88.8	88.0
9	101.7	100.8	99.9	99.0

	109	108	107	106
1	10.9	10.8	10.7	10.6
2	21.8	21.6	21.4	21.2
3	32.7	32.4	32.1	31.8
4	43.6	43.2	42.8	42.4
5	54.5	54.0	53.5	53.0
6	65.4	64.8	64.2	63.6
7	76.3	75.6	74.9	74.2
8	87.2	86.4	85.6	84.8
9	98.1	97.2	96.3	95.4

	105	104	103
1	10.5	10.4	10.3
2	21.0	20.8	20.6
3	31.5	31.2	30.9
4	42.0	41.6	41.2
5	52.5	52.0	51.5
6	63.0	62.4	61.8
7	73.5	72.8	72.1
8	84.0	83.2	82.4
9	94.5	93.6	92.7

P P

7°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	60
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	50
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	40
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	30
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	25
36	9.12 142	94	9.12 525	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	20
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	10
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

P P

	105	104	103
1	10.5	10.4	10.3
2	21.0	20.8	20.6
3	31.5	31.2	30.9
4	42.0	41.6	41.2
5	52.5	52.0	51.5
6	63.0	62.4	61.8
7	73.5	72.8	72.1
8	84.0	83.2	82.4
9	94.5	93.6	92.7

	102	101	99
1	10.2	10.1	9.9
2	20.4	20.2	19.8
3	30.6	30.3	29.7
4	40.8	40.4	39.6
5	51.0	50.5	49.5
6	61.2	60.6	59.4
7	71.4	70.7	69.3
8	81.6	80.8	79.2
9	91.8	90.9	89.1

	98	97	96
1	9.8	9.7	9.6
2	19.6	19.4	19.2
3	29.4	29.1	28.8
4	39.2	38.8	38.4
5	49.0	48.5	48.0
6	58.8	58.2	57.6
7	68.6	67.9	67.2
8	78.4	77.6	76.8
9	88.2	87.3	86.4

	95	94	93
1	9.5	9.4	9.3
2	19.0	18.8	18.6
3	28.5	28.2	27.9
4	38.0	37.6	37.2
5	47.5	47.0	46.5
6	57.0	56.4	55.8
7	66.5	65.8	65.1
8	76.0	75.2	74.4
9	85.5	84.6	83.7

	92	91	90
1	9.2	9.1	9.0
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

P P

82°

8°

'	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	0
	L Cos	d	L Cot	c d	L Tan	L Sin	'

P P

	92	91	90
1	9.2	9.1	9.0
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

	89	88
1	8.9	8.8
2	17.8	17.6
3	26.7	26.4
4	35.6	35.2
5	44.5	44.0
6	53.4	52.8
7	62.3	61.6
8	71.2	70.4
9	80.1	79.2

	87	86	85
1	8.7	8.6	8.5
2	17.4	17.2	17.0
3	26.1	25.8	25.5
4	34.8	34.4	34.0
5	43.5	43.0	42.5
6	52.2	51.6	51.0
7	60.9	60.2	59.5
8	69.6	68.8	68.0
9	78.3	77.4	76.5

	84	83
1	8.4	8.3
2	16.8	16.6
3	25.2	24.9
4	33.6	33.2
5	42.0	41.5
6	50.4	49.8
7	58.8	58.1
8	67.2	66.4
9	75.6	74.7

	82	81	80
1	8.2	8.1	8.0
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

P P

81°

9°

′	L Sin	d	L Tan	c d	L Cot	L Cos	
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	25
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	0.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	0
	L Cos	d	L Cot	c d	L Tan	L Sin	′

P P

	82	81	80
1	8.2	8.1	8.0
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

	79	78	77
1	7.9	7.8	7.7
2	15.8	15.6	15.4
3	23.7	23.4	23.1
4	31.6	31.2	30.8
5	39.5	39.0	38.5
6	47.4	46.8	46.2
7	55.3	54.6	53.9
8	63.2	62.4	61.6
9	71.1	70.2	69.3

	76	75	74
1	7.6	7.5	7.4
2	15.2	15.0	14.8
3	22.8	22.5	22.2
4	30.4	30.0	29.6
5	38.0	37.5	37.0
6	45.6	45.0	44.4
7	53.2	52.5	51.8
8	60.8	60.0	59.2
9	68.4	67.5	66.6

	73	72	71
1	7.3	7.2	7.1
2	14.6	14.4	14.2
3	21.9	21.6	21.3
4	29.2	28.8	28.4
5	36.5	36.0	35.5
6	43.8	43.2	42.6
7	51.1	50.4	49.7
8	58.4	57.6	56.8
9	65.7	64.8	63.9

	3	2
1	0.3	0.2
2	0.6	0.4
3	0.9	0.6
4	1.2	0.8
5	1.5	1.0
6	1.8	1.2
7	2.1	1.4
8	2.4	1.6
9	2.7	1.8

P P

80°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	2	60
1	9.24 039	71	9.24 706	73	0.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	74	0.75 221	9.99 331	3	58
3	9.24 181	72	9.24 853	73	0.75 147	9.99 328	2	57
4	9.24 253	71	9.24 926	74	0.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	73	0.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	3	54
7	9.24 466	70	9.25 146	73	0.74 854	9.99 319	2	53
8	9.24 536	71	9.25 219	73	0.74 781	9.99 317	2	52
9	9.24 607	70	9.25 292	73	0.74 708	9.99 315	2	51
10	9.24 677	71	9.25 365	72	0.74 635	9.99 313	3	50
11	9.24 748	70	9.25 437	73	0.74 563	9.99 310	2	49
12	9.24 818	70	9.25 510	72	0.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	73	0.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	72	0.74 345	9.99 304	3	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	2	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44
17	9.25 168	69	9.25 871	72	0.74 129	9.99 297	3	43
18	9.25 237	70	9.25 943	72	0.74 057	9.99 294	2	42
19	9.25 307	69	9.26 015	71	0.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	71	0.73 842	9.99 288	3	39
22	9.25 514	69	9.26 229	72	0.73 771	9.99 285	2	38
23	9.25 583	69	9.26 301	71	0.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	3	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	2	35
26	9.25 790	68	9.26 514	71	0.73 486	9.99 276	2	34
27	9.25 858	69	9.26 585	70	0.73 415	9.99 274	3	33
28	9.25 927	68	9.26 655	71	0.73 345	9.99 271	2	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	3	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	2	29
32	9.26 199	68	9.26 937	71	0.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	70	0.72 992	9.99 260	3	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	2	26
35	9.26 403	67	9.27 148	70	0.72 852	9.99 255	3	25
36	9.26 470	68	9.27 218	70	0.72 782	9.99 252	2	24
37	9.26 538	67	9.27 288	69	0.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	70	0.72 643	9.99 248	3	22
39	9.26 672	67	9.27 427	69	0.72 573	9.99 245	2	21
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	69	0.72 434	9.99 241	3	19
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	2	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	3	17
44	9.27 007	66	9.27 773	69	0.72 227	9.99 233	2	16
45	9.27 073	67	9.27 842	69	0.72 158	9.99 231	2	15
46	9.27 140	66	9.27 911	69	0.72 089	9.99 229	3	14
47	9.27 206	67	9.27 980	69	0.72 020	9.99 226	2	13
48	9.27 273	66	9.28 049	68	0.71 951	9.99 224	3	12
49	9.27 339	66	9.28 117	69	0.71 883	9.99 221	2	11
50	9.27 405	66	9.28 186	68	0.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	69	0.71 746	9.99 217	3	9
52	9.27 537	65	9.28 323	68	0.71 677	9.99 214	2	8
53	9.27 602	66	9.28 391	68	0.71 609	9.99 212	3	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	2	6
55	9.27 734	65	9.28 527	68	0.71 473	9.99 207	3	5
56	9.27 799	65	9.28 595	67	0.71 405	9.99 204	2	4
57	9.27 864	66	9.28 662	68	0.71 338	9.99 202	2	3
58	9.27 930	65	9.28 730	68	0.71 270	9.99 200	3	2
59	9.27 995	65	9.28 798	67	0.71 202	9.99 197	2	1
60	9.28 060		9.28 865		0.71 135	9.99 195		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	74	73	72
1	7.4	7.3	7.2
2	14.8	14.6	14.4
3	22.2	21.9	21.6
4	29.6	29.2	28.8
5	37.0	36.5	36.0
6	44.4	43.8	43.2
7	51.8	51.1	50.4
8	59.2	58.4	57.6
9	66.6	65.7	64.8

	71	70	69
1	7.1	7.0	6.9
2	14.2	14.0	13.8
3	21.3	21.0	20.7
4	28.4	28.0	27.6
5	35.5	35.0	34.5
6	42.6	42.0	41.4
7	49.7	49.0	48.3
8	56.8	56.0	55.2
9	63.9	63.0	62.1

	68	67
1	6.8	6.7
2	13.6	13.4
3	20.4	20.1
4	27.2	26.8
5	34.0	33.5
6	40.8	40.2
7	47.6	46.9
8	54.4	53.6
9	61.2	60.3

	66	65
1	6.6	6.5
2	13.2	13.0
3	19.8	19.5
4	26.4	26.0
5	33.0	32.5
6	39.6	39.0
7	46.2	45.5
8	52.8	52.0
9	59.4	58.5

	3	2
1	0.3	0.2
2	0.6	0.4
3	0.9	0.6
4	1.2	0.8
5	1.5	1.0
6	1.8	1.2
7	2.1	1.4
8	2.4	1.6
9	2.7	1.8

P P

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

P P

	68	67	66
1	6.8	6.7	6.6
2	13.6	13.4	13.2
3	20.4	20.1	19.8
4	27.2	26.8	26.4
5	34.0	33.5	33.0
6	40.8	40.2	39.6
7	47.6	46.9	46.2
8	54.4	53.6	52.8
9	61.2	60.3	59.4

	65	64	63
1	6.5	6.4	6.3
2	13.0	12.8	12.6
3	19.5	19.2	18.9
4	26.0	25.6	25.2
5	32.5	32.0	31.5
6	39.0	38.4	37.8
7	45.5	44.8	44.1
8	52.0	51.2	50.4
9	58.5	57.6	56.7

	62	61
1	6.2	6.1
2	12.4	12.2
3	18.6	18.3
4	24.8	24.4
5	31.0	30.5
6	37.2	36.6
7	43.4	42.7
8	49.6	48.8
9	55.8	54.9

	60	59
1	6.0	5.9
2	12.0	11.8
3	18.0	17.7
4	24.0	23.6
5	30.0	29.5
6	36.0	35.4
7	42.0	41.3
8	48.0	47.2
9	54.0	53.1

	3	2
1	0.3	0.2
2	0.6	0.4
3	0.9	0.6
4	1.2	0.8
5	1.5	1.0
6	1.8	1.2
7	2.1	1.4
8	2.4	1.6
9	2.7	1.8

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	60
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	2	57
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	3	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	2	50
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	3	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	40
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	3	37
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	3	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	30
31	9.33 591	56	9.34 635	60	0.65 365	9.98 955	2	29
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755	59	0.65 245	9.98 950	3	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	20
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	3	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	10
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	3	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	63	62	61
1	6.3	6.2	6.1
2	12.6	12.4	12.2
3	18.9	18.6	18.3
4	25.2	24.8	24.4
5	31.5	31.0	30.5
6	37.8	37.2	36.6
7	44.1	43.4	42.7
8	50.4	49.6	48.8
9	56.7	55.8	54.9

	60	59
1	6.0	5.9
2	12.0	11.8
3	18.0	17.7
4	24.0	23.6
5	30.0	29.5
6	36.0	35.4
7	42.0	41.3
8	48.0	47.2
9	54.0	53.1

	58	57
1	5.8	5.7
2	11.6	11.4
3	17.4	17.1
4	23.2	22.8
5	29.0	28.5
6	34.8	34.2
7	40.6	39.9
8	46.4	45.6
9	52.2	51.3

	56	55
1	5.6	5.5
2	11.2	11.0
3	16.8	16.5
4	22.4	22.0
5	28.0	27.5
6	33.6	33.0
7	39.2	38.5
8	44.8	44.0
9	50.4	49.5

	3	2
1	0.3	0.2
2	0.6	0.4
3	0.9	0.6
4	1.2	0.8
5	1.5	1.0
6	1.8	1.2
7	2.1	1.4
8	2.4	1.6
9	2.7	1.8

P P

13°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.35 209	54	9.36 336	58	0.63 664	9.98 872	3	60
1	9.35 263	55	9.36 394	58	0.63 606	9.98 869	2	59
2	9.35 318	55	9.36 452	57	0.63 548	9.98 867	3	58
3	9.35 373	54	9.36 509	57	0.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	58	0.63 434	9.98 861	3	56
5	9.35 481	55	9.36 624	57	0.63 376	9.98 858	3	55
6	9.35 536	54	9.36 681	57	0.63 319	9.98 855	3	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	3	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	3	46
15	9.36 022	53	9.37 193	57	0.62 807	9.98 828	3	45
16	9.36 075	54	9.37 250	56	0.62 750	9.98 825	3	44
17	9.36 129	53	9.37 306	57	0.62 694	9.98 822	3	43
18	9.36 182	54	9.37 363	56	0.62 637	9.98 819	3	42
19	9.36 236	53	9.37 419	57	0.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39
22	9.36 395	54	9.37 588	56	0.62 412	9.98 807	3	38
23	9.36 449	53	9.37 644	56	0.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	3	35
26	9.36 608	52	9.37 812	56	0.62 188	9.98 795	3	34
27	9.36 660	53	9.37 868	56	0.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	55	0.62 020	9.98 786	3	31
30	9.36 819	52	9.38 035	56	0.61 965	9.98 783	3	30
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	3	29
32	9.36 924	52	9.38 147	55	0.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27
34	9.37 028	53	9.38 257	56	0.61 743	9.98 771	3	26
35	9.37 081	52	9.38 313	55	0.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	3	24
37	9.37 185	52	9.38 423	56	0.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	4	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	3	18
43	9.37 497	52	9.38 754	54	0.61 246	9.98 743	3	17
44	9.37 549	51	9.38 808	55	0.61 192	9.98 740	3	16
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	3	15
46	9.37 652	51	9.38 918	54	0.61 082	9.98 734	3	14
47	9.37 703	52	9.38 972	55	0.61 028	9.98 731	3	13
48	9.37 755	51	9.39 027	55	0.60 973	9.98 728	3	12
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725	3	11
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	4	9
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	3	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	4	1
60	9.38 368		9.39 677		0.60 323	9.98 690		0
	L Cos	P	L Cot	c d	L Tan	L Sin	d	

P P

	58	57
1	5.8	5.7
2	11.6	11.4
3	17.4	17.1
4	23.2	22.8
5	29.0	28.5
6	34.8	34.2
7	40.6	39.9
8	46.4	45.6
9	52.2	51.3

	56	55
1	5.6	5.5
2	11.2	11.0
3	16.8	16.5
4	22.4	22.0
5	28.0	27.5
6	33.6	33.0
7	39.2	38.5
8	44.8	44.0
9	50.4	49.5

	54	53
1	5.4	5.3
2	10.8	10.6
3	16.2	15.9
4	21.6	21.2
5	27.0	26.5
6	32.4	31.8
7	37.8	37.1
8	43.2	42.4
9	48.6	47.7

	52	51
1	5.2	5.1
2	10.4	10.2
3	15.6	15.3
4	20.8	20.4
5	26.0	25.5
6	31.2	30.6
7	36.4	35.7
8	41.6	40.8
9	46.8	45.9

		3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

P P

76°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	50	9.39 785	53	0.60 215	9.98 684	3	58
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	54	0.60 055	9.98 675	4	55
6	9.38 670	51	9.39 999	53	0.60 001	9.98 671	3	54
7	9.38 721	50	9.40 052	54	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	53	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	54	0.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	53	0.59 734	9.98 656	4	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	3	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425	53	0.59 575	9.98 646	3	46
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	4	44
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	3	43
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	4	40
21	9.39 418	49	9.40 795	52	0.59 205	9.98 623	3	39
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	4	36
25	9.39 615	49	9.41 005	52	0.58 995	9.98 610	3	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	4	32
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	3	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	4	28
33	9.40 006	49	9.41 422	52	0.58 578	9.98 584	3	27
34	9.40 055	48	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	4	25
36	9.40 152	48	9.41 578	51	0.58 422	9.98 574	3	24
37	9.40 200	49	9.41 629	52	0.58 371	9.98 571	3	23
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	49	9.41 733	51	0.58 267	9.98 565	4	21
40	9.40 346	48	9.41 784	52	0.58 216	9.98 561	3	20
41	9.40 394	48	9.41 836	51	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	52	0.58 113	9.98 555	4	18
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	3	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	52	0.57 959	9.98 545	4	15
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	3	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805	9.98 535	4	12
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	3	11
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525	4	9
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	3	8
53	9.40 968	48	9.42 450	51	0.57 550	9.98 518	3	7
54	9.41 016	47	9.42 501	51	0.57 499	9.98 515	4	6
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	3	5
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	51	0.57 347	9.98 505	4	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	3	2
59	9.41 252	48	9.42 755	50	0.57 245	9.98 498	4	1
60	9.41 300		9.42 805		0.57 195	9.98 494		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	54	53
1	5.4	5.3
2	10.8	10.6
3	16.2	15.9
4	21.6	21.2
5	27.0	26.5
6	32.4	31.8
7	37.8	37.1
8	43.2	42.4
9	48.6	47.7

	52	51
1	5.2	5.1
2	10.4	10.2
3	15.6	15.3
4	20.8	20.4
5	26.0	25.5
6	31.2	30.6
7	36.4	35.7
8	41.6	40.8
9	46.8	45.9

	50	49
1	5.0	4.9
2	10.0	9.8
3	15.0	14.7
4	20.0	19.6
5	25.0	24.5
6	30.0	29.4
7	35.0	34.3
8	40.0	39.2
9	45.0	44.1

	48	47
1	4.8	4.7
2	9.6	9.4
3	14.4	14.1
4	19.2	18.8
5	24.0	23.5
6	28.8	28.2
7	33.6	32.9
8	38.4	37.6
9	43.2	42.3

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

75°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.41 300	47	9.42 805	51	0.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	0.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 750		0.54 250	9.98 284		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	51	50	49	48	47	46	45	44	4	3
1	5.1	5.0	4.9	4.8	4.7	4.6	4.5	4.4	0.4	0.3
2	10.2	10.0	9.8	9.6	9.4	9.2	9.0	8.8	0.8	0.6
3	15.3	15.0	14.7	14.4	14.1	13.8	13.5	13.2	1.2	0.9
4	20.4	20.0	19.6	19.2	18.8	18.4	18.0	17.6	1.6	1.2
5	25.5	25.0	24.5	24.0	23.5	23.0	22.5	22.0	2.0	1.5
6	30.6	30.0	29.4	28.8	28.2	27.6	27.0	26.4	2.4	1.8
7	35.7	35.0	34.3	33.6	32.9	32.2	31.5	30.8	2.8	2.1
8	40.8	40.0	39.2	38.4	37.6	36.8	36.0	35.2	3.2	2.4
9	45.9	45.0	44.1	43.2	42.3	41.4	40.5	39.6	3.6	2.7

P P

16°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	60
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	4	59
2	9.44 122	44	9.45 845	47	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	3	57
4	9.44 210	43	9.45 940	47	0.54 060	9.98 270	4	56
5	9.44 253	44	9.45 987	48	0.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	47	0.53 965	9.98 262	3	54
7	9.44 341	44	9.46 082	48	0.53 918	9.98 259	4	53
8	9.44 385	43	9.46 130	47	0.53 870	9.98 255	4	52
9	9.44 428	44	9.46 177	47	0.53 823	9.98 251	3	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	4	50
11	9.44 516	43	9.46 271	48	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	47	0.53 681	9.98 240	3	48
13	9.44 602	44	9.46 366	47	0.53 634	9.98 237	4	47
14	9.44 646	43	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	44	9.46 460	47	0.53 540	9.98 229	3	45
16	9.44 733	43	9.46 507	47	0.53 493	9.98 226	4	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	3	42
19	9.44 862	43	9.46 648	46	0.53 352	9.98 215	4	41
20	9.44 905	43	9.46 694	47	0.53 306	9.98 211	4	40
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	3	39
22	9.44 992	43	9.46 788	47	0.53 212	9.98 204	4	38
23	9.45 035	42	9.46 835	46	0.53 165	9.98 200	4	37
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	3	35
26	9.45 163	43	9.46 975	46	0.53 025	9.98 189	4	34
27	9.45 206	43	9.47 021	47	0.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	46	0.52 932	9.98 181	4	32
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	3	31
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	4	30
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	3	27
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	4	26
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155	4	25
36	9.45 589	43	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	42	9.47 484	46	0.52 516	9.98 147	3	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	4	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	20
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	3	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	4	18
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	3	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	4	13
48	9.46 095	41	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	42	9.48 035	45	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	3	8
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	4	7
54	9.46 345	41	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	5
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	3	1
60	9.46 594		9.48 534		0.51 466	9.98 060		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	48	47
1	4.8	4.7
2	9.6	9.4
3	14.4	14.1
4	19.2	18.8
5	24.0	23.5
6	28.8	28.2
7	33.6	32.9
8	38.4	37.6
9	43.2	42.3

	46	45
1	4.6	4.5
2	9.2	9.0
3	13.8	13.5
4	18.4	18.0
5	23.0	22.5
6	27.6	27.0
7	32.2	31.5
8	36.8	36.0
9	41.4	40.5

	44	43
1	4.4	4.3
2	8.8	8.6
3	13.2	12.9
4	17.6	17.2
5	22.0	21.5
6	26.4	25.8
7	30.8	30.1
8	35.2	34.4
9	39.6	38.7

	42	41
1	4.2	4.1
2	8.4	8.2
3	12.6	12.3
4	16.8	16.4
5	21.0	20.5
6	25.2	24.6
7	29.4	28.7
8	33.6	32.8
9	37.8	36.9

	4	3
1	0.4	0.3
2	0.8	0.6
3	1.2	0.9
4	1.6	1.2
5	2.0	1.5
6	2.4	1.8
7	2.8	2.1
8	3.2	2.4
9	3.6	2.7

P P

73°

17°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	45	44	43
1	4.5	4.4	4.3
2	9.0	8.8	8.6
3	13.5	13.2	12.9
4	18.0	17.6	17.2
5	22.5	22.0	21.5
6	27.0	26.4	25.8
7	31.5	30.8	30.1
8	36.0	35.2	34.4
9	40.5	39.6	38.7

	42	41
1	4.2	4.1
2	8.4	8.2
3	12.6	12.3
4	16.8	16.4
5	21.0	20.5
6	25.2	24.6
7	29.4	28.7
8	33.6	32.8
9	37.8	36.9

	40	39
1	4.0	3.9
2	8.0	7.8
3	12.0	11.7
4	16.0	15.6
5	20.0	19.5
6	24.0	23.4
7	28.0	27.3
8	32.0	31.2
9	36.0	35.1

	5	4	3
1	0.5	0.4	0.3
2	1.0	0.8	0.6
3	1.5	1.2	0.9
4	2.0	1.6	1.2
5	2.5	2.0	1.5
6	3.0	2.4	1.8
7	3.5	2.8	2.1
8	4.0	3.2	2.4
9	4.5	3.6	2.7

P P

72°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	5	59
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	4	58
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	39	9.51 478	42	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	43	0.48 480	9.97 788	4	52
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	5	51
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	4	50
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	5	45
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	4	44
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	4	25
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8
53	9.51 007	36	9.53 409	41	0.46 591	9.97 597	4	7
54	9.51 043	37	9.53 450	42	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1
60	9.51 264		9.53 697		0.46 303	9.97 567		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	43	42	41
1	4.3	4.2	4.1
2	8.6	8.4	8.2
3	12.9	12.6	12.3
4	17.2	16.8	16.4
5	21.5	21.0	20.5
6	25.8	25.2	24.6
7	30.1	29.4	28.7
8	34.4	33.6	32.8
9	38.7	37.8	36.9

	39	38
1	3.9	3.8
2	7.8	7.6
3	11.7	11.4
4	15.6	15.2
5	19.5	19.0
6	23.4	22.8
7	27.3	26.6
8	31.2	30.4
9	35.1	34.2

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

P P

71°

19°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	60
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	4	50
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	40
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	5	30
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	4	25
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	4	21
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	5	20
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	10
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	41	40	39
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1

	37	36
1	3.7	3.6
2	7.4	7.2
3	11.1	10.8
4	14.8	14.4
5	18.5	18.0
6	22.2	21.6
7	25.9	25.2
8	29.6	28.8
9	33.3	32.4

	35	34
1	3.5	3.4
2	7.0	6.8
3	10.5	10.2
4	14.0	13.6
5	17.5	17.0
6	21.0	20.4
7	24.5	23.8
8	28.0	27.2
9	31.5	30.6

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

P P

70°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475	34	9.56 185	39	0.43 815	9.97 289	4	58
3	9.53 509	35	9.56 224	40	0.43 776	9.97 285	5	57
4	9.53 544	34	9.56 264	39	0.43 736	9.97 280	4	56
5	9.53 578	35	9.56 303	39	0.43 697	9.97 276	5	55
6	9.53 613	34	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	35	9.56 381	39	0.43 619	9.97 266	4	53
8	9.53 682	34	9.56 420	39	0.43 580	9.97 262	5	52
9	9.53 716	35	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	34	9.56 498	39	0.43 502	9.97 252	4	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	5	49
12	9.53 819	35	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	34	9.56 615	39	0.43 385	9.97 238	4	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	5	46
15	9.53 922	35	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	34	9.56 732	39	0.43 268	9.97 224	4	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	5	43
18	9.54 025	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	4	41
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	5	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965	39	0.43 035	9.97 196	4	38
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	5	37
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	4	35
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	5	34
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33
28	9.54 365	34	9.57 197	38	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	39	0.42 765	9.97 163	4	31
30	9.54 433	33	9.57 274	38	0.42 726	9.97 159	5	30
31	9.54 466	34	9.57 312	39	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	4	28
33	9.54 534	33	9.57 389	39	0.42 611	9.97 145	5	27
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635	33	9.57 504	39	0.42 496	9.97 130	4	24
37	9.54 668	34	9.57 543	38	0.42 457	9.97 126	5	23
38	9.54 702	33	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	34	9.57 619	39	0.42 381	9.97 116	5	21
40	9.54 769	33	9.57 658	38	0.42 342	9.97 111	4	20
41	9.54 802	34	9.57 696	38	0.42 304	9.97 107	5	19
42	9.54 836	33	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	34	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	33	9.57 810	39	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	38	0.42 151	9.97 087	4	15
46	9.54 969	34	9.57 887	38	0.42 113	9.97 083	5	14
47	9.55 003	33	9.57 925	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	34	9.58 039	38	0.41 961	9.97 063	4	10
51	9.55 136	33	9.58 077	38	0.41 923	9.97 059	5	9
52	9.55 169	33	9.58 115	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	4	5
56	9.55 301	33	9.58 267	37	0.41 733	9.97 035	5	4
57	9.55 334	33	9.58 304	38	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433		9.58 418		0.41 582	9.97 015		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	40	39	38
1	4.0	3.9	3.8
2	8.0	7.8	7.6
3	12.0	11.7	11.4
4	16.0	15.6	15.2
5	20.0	19.5	19.0
6	24.0	23.4	22.8
7	28.0	27.3	26.6
8	32.0	31.2	30.4
9	36.0	35.1	34.2

	37	35
1	3.7	3.5
2	7.4	7.0
3	11.1	10.5
4	14.8	14.0
5	18.5	17.5
6	22.2	21.0
7	25.9	24.5
8	29.6	28.0
9	33.3	31.5

	34	33
1	3.4	3.3
2	6.8	6.6
3	10.2	9.9
4	13.6	13.2
5	17.0	16.5
6	20.4	19.8
7	23.8	23.1
8	27.2	26.4
9	30.6	29.7

	5	4
1	0.5	0.4
2	1.0	0.8
3	1.5	1.2
4	2.0	1.6
5	2.5	2.0
6	3.0	2.4
7	3.5	2.8
8	4.0	3.2
9	4.5	3.6

P P

69°

21°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	0.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 704	38	0.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.49 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	0.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9 59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0 39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9 60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9 60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9 60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	38	37	36
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4

	33	32	31
1	3.3	3.2	3.1
2	6.6	6.4	6.2
3	9.9	9.6	9.3
4	13.2	12.8	12.4
5	16.5	16.0	15.5
6	19.8	19.2	18.6
7	23.1	22.4	21.7
8	26.4	25.6	24.8
9	29.7	28.8	27.9

	6	5	4
1	0.6	0.5	0.4
2	1.2	1.0	0.8
3	1.8	1.5	1.2
4	2.4	2.0	1.6
5	3.0	2.5	2.0
6	3.6	3.0	2.4
7	4.2	3.5	2.8
8	4.8	4.0	3.2
9	5.4	4.5	3.6

P P

68°

22°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	37	36	35
1	3.7	3.6	3.5
2	7.4	7.2	7.0
3	11.1	10.8	10.5
4	14.8	14.4	14.0
5	18.5	18.0	17.5
6	22.2	21.6	21.0
7	25.9	25.2	24.5
8	29.6	28.8	28.0
9	33.3	32.4	31.5

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	30	29
1	3.0	2.9
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

67°

23°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.59 188	30	9.62 785	35	0.37 215	9.96 403	6	60
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865	9.96 349	6	50
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175	34	0.35 825	9.96 185	6	20
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	0.35 448	9.66 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	36	35	34
1	3.6	3.5	3.4
2	7.2	7.0	6.8
3	10.8	10.5	10.2
4	14.4	14.0	13.6
5	18.0	17.5	17.0
6	21.6	21.0	20.4
7	25.2	24.5	23.8
8	28.8	28.0	27.2
9	32.4	31.5	30.6

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	6	5
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

P P

66°

24°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	**60**
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	**59**
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	6	**58**
3	9.61 016	29	9.64 960	34	0.35 040	9.96 056	6	**57**
4	9.61 045	28	9.64 994	34	0.35 006	9.96 050	5	**56**
5	9.61 073	28	9.65 028	34	0.34 972	9.96 045	6	**55**
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	5	**54**
7	9.61 129	29	9.65 096	34	0.34 904	9.96 034	6	**53**
8	9.61 158	28	9.65 130	34	0.34 870	9.96 028	6	**52**
9	9.61 186	28	9.65 164	33	0.34 836	9.96 022	5	**51**
10	9.61 214	28	9.65 197	34	0.34 803	9.96 017	6	**50**
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	**49**
12	9.61 270	28	9.65 265	34	0.34 735	9.96 005	5	**48**
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	6	**47**
14	9.61 326	28	9.65 333	33	0.34 667	9.95 994	6	**46**
15	9.61 354	28	9.65 366	34	0.34 634	9.95 988	6	**45**
16	9.61 382	29	9.65 400	34	0.34 600	9.95 982	5	**44**
17	9.61 411	27	9.65 434	33	0.34 566	9.95 977	6	**43**
18	9.61 438	28	9.65 467	34	0.34 533	9.95 971	6	**42**
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	**41**
20	9.61 494	28	9.65 535	33	0.34 465	9.95 960	6	**40**
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	6	**39**
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	**38**
23	9.61 578	28	9.65 636	33	0.34 364	9.95 942	5	**37**
24	9.61 606	28	9.65 669	34	0.34 331	9.95 937	6	**36**
25	9.61 634	28	9.65 703	33	0.34 297	9.95 931	6	**35**
26	9.61 662	27	9.65 736	34	0.34 264	9.95 925	5	**34**
27	9.61 689	28	9.65 770	33	0.34 230	9.95 920	6	**33**
28	9.61 717	28	9.65 803	34	0.34 197	9.95 914	6	**32**
29	9.61 745	28	9.65 837	33	0.34 163	9.95 908	6	**31**
30	9.61 773	27	9.65 870	34	0.34 130	9.95 902	5	**30**
31	9.61 800	28	9.65 904	33	0.34 096	9.95 897	6	**29**
32	9.61 828	28	9.65 937	34	0.34 063	9.95 891	6	**28**
33	9.61 856	27	9.65 971	33	0.34 029	9.95 885	6	**27**
34	9.61 883	28	9.66 004	34	0.33 996	9.95 879	6	**26**
35	9.61 911	28	9.66 038	33	0.33 962	9.95 873	5	**25**
36	9.61 939	27	9.66 071	33	0.33 929	9.95 868	6	**24**
37	9.61 966	28	9.66 104	34	0.33 896	9.95 862	6	**23**
38	9.61 994	27	9.66 138	33	0.33 862	9.95 856	6	**22**
39	9.62 021	28	9.66 171	33	0.33 829	9.95 850	6	**21**
40	9.62 049	27	9.66 204	34	0.33 796	9.95 844	5	**20**
41	9.62 076	28	9.66 238	33	0.33 762	9.95 839	6	**19**
42	9.62 104	27	9.66 271	33	0.33 729	9.95 833	6	**18**
43	9.62 131	28	9.66 304	33	0.33 696	9.95 827	6	**17**
44	9.62 159	27	9.66 337	34	0.33 663	9.95 821	6	**16**
45	9.62 186	28	9.66 371	33	0.33 629	9.95 815	5	**15**
46	9.62 214	27	9.66 404	33	0.33 596	9.95 810	6	**14**
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	**13**
48	9.62 268	28	9.66 470	33	0.33 530	9.95 798	6	**12**
49	9.62 296	27	9.66 503	34	0.33 497	9.95 792	6	**11**
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	**10**
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	**9**
52	9.62 377	28	9.66 603	33	0.33 397	9.95 775	6	**8**
53	9.62 405	27	9.66 636	33	0.33 364	9.95 769	6	**7**
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	**6**
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	**5**
56	9.62 486	27	9.66 735	33	0.33 265	9.95 751	6	**4**
57	9.62 513	28	9.66 768	33	0.33 232	9.95 745	6	**3**
58	9.62 541	27	9.66 801	33	0.33 199	9.95 739	6	**2**
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	**1**
60	9.62 595		9.66 867		0.33 133	9.95 728		**0**
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	34	**33**
1	3.4	3.3
2	6.8	6.6
3	10.2	9.9
4	13.6	13.2
5	17.0	16.5
6	20.4	19.8
7	23.8	23.1
8	27.2	26.4
9	30.6	29.7

	29	**28**	**27**
1	2.9	2.8	2.7
2	5.8	5.6	5.4
3	8.7	8.4	8.1
4	11.6	11.2	10.8
5	14.5	14.0	13.5
6	17.4	16.8	16.2
7	20.3	19.6	18.9
8	23.2	22.4	21.6
9	26.1	25.2	24.3

	6	**5**
1	0.6	0.5
2	1.2	1.0
3	1.8	1.5
4	2.4	2.0
5	3.0	2.5
6	3.6	3.0
7	4.2	3.5
8	4.8	4.0
9	5.4	4.5

65°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.62 595	27	9.66 867	33	0.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	5	50
11	9.62 892	26	9.67 229	33	0.32 771	9.95 663	6	49
12	9.62 918	27	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	32	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	33	0.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	33	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	32	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	33	0.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	32	0.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	33	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	9.63 425	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	33	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	32	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	33	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	33	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	0.31 535	9.95 434	7	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	6	10
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	33	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	32	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	7	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	6	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184		9.68 818		0.31 182	9.95 366		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	33	32
1	3.3	3.2
2	6.6	6.4
3	9.9	9.6
4	13.2	12.8
5	16.5	16.0
6	19.8	19.2
7	23.1	22.4
8	26.4	25.6
9	29.7	28.8

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	7	6	5
1	0.7	0.6	0.5
2	1.4	1.2	1.0
3	2.1	1.8	1.5
4	2.8	2.4	2.0
5	3.5	3.0	2.5
6	4.2	3.6	3.0
7	4.9	4.2	3.5
8	5.6	4.8	4.0
9	6.3	5.4	4.5

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	60
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	50
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	6	46
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42
19	9.64 673	25	9.69 425	32	0.30 575	9.95 248	6	41
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	40
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38
23	9.64 775	25	9.69 552	32	0.30 448	9.95 223	6	37
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36
25	9.64 826	25	9.69 615	32	0.30 385	9.95 211	7	35
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	30
31	9.64 978	25	9.69 805	32	0.30 195	9.95 173	6	29
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24
37	9.65 130	25	9.69 995	31	0.30 005	9.95 135	6	23
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	7	22
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	20
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	6	16
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13
48	9.65 406	25	9.70 341	31	0.29 659	9.95 065	6	12
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	10
51	9.65 481	25	9.70 435	31	0.29 565	9.95 046	7	9
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4
57	9.65 630	25	9.70 623	31	0.29 377	9.95 007	6	3
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2
59	9.65 680	25	9.70 685	32	0.29 315	9.94 995	7	1
60	9.65 705		9.70 717		0.29 283	9.94 988		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	32	31
1	3.2	3.1
2	6.4	6.2
3	9.6	9.3
4	12.8	12.4
5	16.0	15.5
6	19.2	18.6
7	22.4	21.7
8	25.6	24.8
9	28.8	27.9

	26	25	24
1	2.6	2.5	2.4
2	5.2	5.0	4.8
3	7.8	7.5	7.2
4	10.4	10.0	9.6
5	13.0	12.5	12.0
6	15.6	15.0	14.4
7	18.2	17.5	16.8
8	20.8	20.0	19.2
9	23.4	22.5	21.6

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

P P

27°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.65 705	24	9.70 717	31	0.29 283	9.94 988	6	60
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	6	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57
4	9.65 804	24	9.70 841	32	0.29 159	9.94 962	6	56
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	6	54
7	9.65 878	24	9.70 935	31	0.29 065	9.94 943	7	53
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	6	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	6	50
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48
13	9.66 025	25	9.71 121	32	0.28 879	9.94 904	6	47
14	9.66 050	25	9.71 153	31	0.28 847	9.94 898	7	46
15	9.66 075	24	9.71 184	31	0.28 816	9.94 891	6	45
16	9.66 099	25	9.71 215	31	0.28 785	9.94 885	7	44
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	6	42
19	9.66 173	24	9.71 308	31	0.28 692	9.94 865	7	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	6	40
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39
22	9.66 246	24	9.71 401	30	0.28 599	9.94 845	6	38
23	9.66 270	25	9.71 431	31	0.28 569	9.94 839	7	37
24	9.66 295	24	9.71 462	31	0.28 538	9.94 832	6	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	6	34
27	9.66 368	24	9.71 555	31	0.28 445	9.94 813	7	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	6	31
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	30
31	9.66 465	24	9.71 679	30	0.28 321	9.94 786	6	29
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	6	27
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	30	0.28 167	9.94 753	6	24
37	9.66 610	24	9.71 863	31	0.28 137	9.94 747	7	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	6	22
39	9.66 658	24	9.71 925	30	0.28 075	9.94 734	7	21
40	9.66 682	24	9.71 955	31	0.28 045	9.94 727	7	20
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	6	19
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18
43	9.66 755	24	9.72 048	30	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	31	0.27 922	9.94 700	6	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15
46	9.66 827	24	9.72 140	30	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	31	0.27 830	9.94 680	6	13
48	9.66 875	24	9.72 201	30	0.27 799	9.94 674	7	12
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	6	10
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	30	0.27 646	9.94 640	6	7
54	9.67 018	24	9.72 384	31	0.27 616	9.94 634	7	6
55	9.67 042	24	9.72 415	30	0.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	31	0.27 555	9.94 620	6	4
57	9.67 090	23	9.72 476	30	0.27 524	9.94 614	7	3
58	9.67 113	24	9.72 506	31	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	30	0.27 463	9.94 600	7	1
60	9.67 161		9.72 567		0.27 433	9.94 593		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	32	31	30
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

	25	24	23
1	2.5	2.4	2.3
2	5.0	4.8	4.6
3	7.5	7.2	6.9
4	10.0	9.6	9.2
5	12.5	12.0	11.5
6	15.0	14.4	13.8
7	17.5	16.8	16.1
8	20.0	19.2	18.4
9	22.5	21.6	20.7

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

P P

62°

28°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60
1	9.67 185	23	9.72 598	30	0.27 402	9.94 587	7	59
2	9.67 208	24	9.72 628	31	0.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57
4	9.67 256	24	9.72 689	31	0.27 311	9.94 567	7	56
5	9.67 280	23	9.72 720	30	0.27 280	9.94 560	7	55
6	9.67 303	24	9.72 750	30	0.27 250	9.94 553	7	54
7	9.67 327	23	9.72 780	31	0.27 220	9.94 546	6	53
8	9.67 350	24	9.72 811	30	0.27 189	9.94 540	7	52
9	9.67 374	24	9.72 841	31	0.27 159	9.94 533	7	51
10	9.67 398	23	9.72 872	30	0.27 128	9.94 526	7	50
11	9.67 421	24	9.72 902	30	0.27 098	9.94 519	6	49
12	9.67 445	23	9.72 932	31	0.27 068	9.94 513	7	48
13	9.67 468	24	9.72 963	30	0.27 037	9.94 506	7	47
14	9.67 492	23	9.72 993	30	0.27 007	9.94 499	7	46
15	9.67 515	24	9.73 023	31	0.26 977	9.94 492	7	45
16	9.67 539	23	9.73 054	30	0.26 946	9.94 485	6	44
17	9.67 562	24	9.73 084	30	0.26 916	9.94 479	7	43
18	9.67 586	23	9.73 114	30	0.26 886	9.94 472	7	42
19	9.67 609	24	9.73 144	31	0.26 856	9.94 465	7	41
20	9.67 633	23	9.73 175	80	0.26 825	9.94 458	7	40
21	9.67 656	24	9.73 205	30	0.26 795	9.94 451	6	39
22	9.67 680	23	9.73 235	30	0.26 765	9.94 445	7	38
23	9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37
24	9.67 726	24	9.73 295	31	0.26 705	9.94 431	7	36
25	9.67 750	23	9.73 326	30	0.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
27	9.67 796	24	9.73 386	30	0.26 614	9.94 410	6	33
28	9.67 820	23	9.73 416	30	0.26 584	9.94 404	7	32
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
30	9.67 866	24	9.73 476	31	0.26 524	9.94 390	7	30
31	9.67 890	23	9.73 507	30	0.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
35	9.67 982	24	9.73 627	30	0.26 373	9.94 355	6	25
36	9.68 006	23	9.73 657	30	0.26 343	9.94 349	7	24
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
45	9.68 213	24	9.73 927	30	0.26 073	9.94 286	7	15
46	9.68 237	23	9.73 957	30	0.26 043	9.94 279	6	14
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	7	13
48	9.68 283	22	9.74 017	30	0.25 983	9.94 266	7	12
49	9.68 305	23	9.74 047	30	0.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
58	9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2
59	9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1
60	9.68 557		9.74 375		0.25 625	9.94 182		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	31	30	29
1	3.1	3.0	2.9
2	6.2	6.0	5.8
3	9.3	9.0	8.7
4	12.4	12.0	11.6
5	15.5	15.0	14.5
6	18.6	18.0	17.4
7	21.7	21.0	20.3
8	24.8	24.0	23.2
9	27.9	27.0	26.1

	24	23	22
1	2.4	2.3	2.2
2	4.8	4.6	4.4
3	7.2	6.9	6.6
4	9.6	9.2	8.8
5	12.0	11.5	11.0
6	14.4	13.8	13.2
7	16.8	16.1	15.4
8	19.2	18.4	17.6
9	21.6	20.7	19.8

	7	6
1	0.7	0.6
2	1.4	1.2
3	2.1	1.8
4	2.8	2.4
5	3.5	3.0
6	4.2	3.6
7	4.9	4.2
8	5.6	4.8
9	6.3	5.4

P P

61°

29°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	60
1	9.68 580	23	9.74 405	30	0.25 595	9.94 175	7	59
2	9.68 603	22	9.74 435	30	0.25 565	9.94 168	7	58
3	9.68 625	23	9.74 465	29	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	50
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105	7	49
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	8	48
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47
14	9.68 875	22	9.74 791	30	0.25 209	9.94 083	7	46
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	45
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43
18	9.68 965	22	9.74 910	29	0.25 090	9.94 055	7	42
19	9.68 987	23	9.74 939	30	0.25 061	9.94 048	7	41
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	40
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39
22	9.69 055	22	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	8	37
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	35
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33
28	9.69 189	23	9.75 205	30	0.24 795	9.93 984	7	32
29	9.69 212	22	9.75 235	29	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	30
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	8	29
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	25
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	8	23
38	9.69 412	22	9.75 500	29	0.24 500	9.93 912	7	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	20
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	8	18
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	30	0.24 295	9.93 862	7	15
46	9.69 589	22	9.75 735	29	0.24 265	9.93 855	8	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	10
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	8	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	8	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	5
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	8	2
59	9.69 875	22	9.76 115	29	0.23 885	9.93 760	7	1
60	9.69 897		9.76 144		0.23 856	9.93 753		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	30	29
1	3.0	2.9
2	6.0	5.8
3	9.0	8.7
4	12.0	11.6
5	15.0	14.5
6	18.0	17.4
7	21.0	20.3
8	24.0	23.2
9	27.0	26.1

	23	22
1	2.3	2.2
2	4.6	4.4
3	6.9	6.6
4	9.2	8.8
5	11.5	11.0
6	13.8	13.2
7	16.1	15.4
8	18.4	17.6
9	20.7	19.8

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

P P

60°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	8	47
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	45
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	8	39
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	20
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9.93 427	7	16
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	8	15
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	30	29	28
1	3.0	2.9	2.8
2	6.0	5.8	5.6
3	9.0	8.7	8.4
4	12.0	11.6	11.2
5	15.0	14.5	14.0
6	18.0	17.4	16.8
7	21.0	20.3	19.6
8	24.0	23.2	22.4
9	27.0	26.1	25.2

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

P P

31°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	′
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	60
1	9.71 205	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	20
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	21	20
1	2.1	2.0
2	4.2	4.0
3	6.3	6.0
4	8.4	8.0
5	10.5	10.0
6	12.6	12.0
7	14.7	14.0
8	16.8	16.0
9	18.9	18.0

	8	7
1	0.8	0.7
2	1.6	1.4
3	2.4	2.1
4	3.2	2.8
5	4.0	3.5
6	4.8	4.2
7	5.6	4.9
8	6.4	5.6
9	7.2	6.3

58°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	29	28	27
1	2.9	2.8	2.7
2	5.8	5.6	5.4
3	8.7	8.4	8.1
4	11.6	11.2	10.8
5	14.5	14.0	13.5
6	17.4	16.8	16.2
7	20.3	19.6	18.9
8	23.2	22.4	21.6
9	26.1	25.2	24.3

	21	20	19
1	2.1	2.0	1.9
2	4.2	4.0	3.8
3	6.3	6.0	5.7
4	8.4	8.0	7.6
5	10.5	10.0	9.5
6	12.6	12.0	11.4
7	14.7	14.0	13.3
8	16.8	16.0	15.2
9	18.9	18.0	17.1

	9	8	7
1	0.9	0.8	0.7
2	1.8	1.6	1.4
3	2.7	2.4	2.1
4	3.6	3.2	2.8
5	4.5	4.0	3.5
6	5.4	4.8	4.2
7	6.3	5.6	4.9
8	7.2	6.4	5.6
9	8.1	7.2	6.3

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	60
1	9.73 630	20	9.81 279	28	0.18 721	9.92 351	8	59
2	9.73 650	19	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	50
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	45
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	40
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	35
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	30
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	9	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	8	26
35	9.74 284	19	9.82 215	28	0.17 785	9.92 069	9	25
36	9.74 303	19	9.82 243	27	0.17 757	9.92 060	8	24
37	9.74 322	19	9.82 270	28	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	27	0.17 702	9.92 044	9	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	8	21
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	9	20
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	8	19
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	9	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	8	16
45	9.74 474	19	9.82 489	28	0.17 511	9.91 985	9	15
46	9.74 493	19	9.82 517	27	0.17 483	9.91 976	8	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	9	13
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	8	12
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	9	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	8	10
51	9.74 587	19	9.82 653	28	0.17 347	9.91 934	9	9
52	9.74 606	19	9.82 681	27	0.17 319	9.91 925	8	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	9	7
54	9.74 644	18	9.82 735	27	0.17 265	9.91 908	8	6
55	9.74 662	19	9.82 762	28	0.17 238	9.91 900	9	5
56	9.74 681	19	9.82 790	27	0.17 210	9.91 891	8	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	9	3
58	9.74 719	18	9.82 844	27	0.17 156	9.91 874	8	2
59	9.74 737	19	9.82 871	28	0.17 129	9.91 866	9	1
60	9.74 756		9.82 899		0.17 101	9.91 857		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	28	27
1	2.8	2.7
2	5.6	5.4
3	8.4	8.1
4	11.2	10.8
5	14.0	13.5
6	16.8	16.2
7	19.6	18.9
8	22.4	21.6
9	25.2	24.3

	20	19	18
1	2.0	1.9	1.8
2	4.0	3.8	3.6
3	6.0	5.7	5.4
4	8.0	7.6	7.2
5	10.0	9.5	9.0
6	12.0	11.4	10.8
7	14.0	13.3	12.6
8	16.0	15.2	14.4
9	18.0	17.1	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	60
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	50
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	40
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	30
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	20
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	10
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	28	27	26
1	2.8	2.7	2.6
2	5.6	5.4	5.2
3	8.4	8.1	7.8
4	11.2	10.8	10.4
5	14.0	13.5	13.0
6	16.8	16.2	15.6
7	19.6	18.9	18.2
8	22.4	21.6	20.8
9	25.2	24.3	23.4

	19	18
1	1.9	1.8
2	3.8	3.6
3	5.7	5.4
4	7.6	7.2
5	9.5	9.0
6	11.4	10.8
7	13.3	12.6
8	15.2	14.4
9	17.1	16.2

	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	9.75 877	18	9.84 550	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075	18	9.84 845	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075	9.91 203	9	45
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695	17	9.85 780	27	0.14 220	9.90 915	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	9.76 765	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835	17	9.85 993	27	0.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17
1	1.8	1.7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3

	10	9	8
1	1.0	0.9	0.8
2	2.0	1.8	1.6
3	3.0	2.7	2.4
4	4.0	3.6	3.2
5	5.0	4.5	4.0
6	6.0	5.4	4.8
7	7.0	6.3	5.6
8	8.0	7.2	6.4
9	9.0	8.1	7.2

P P

54°

36°

′	L Sin	d	L Tan	cd	L Cot	L Cos	d	
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	27	0.13 715	9.90 741	10	54
7	9.77 043	18	9.86 312	26	0.13 688	9.90 731	9	53
8	9.77 061	17	9.86 338	27	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	10	50
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	9	49
12	9.77 130	17	9.86 445	26	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	10	46
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	9	45
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	26	0.13 317	9.90 602	10	39
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	9	38
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565	10	35
26	9.77 370	17	9.86 815	27	0.13 185	9.90 555	9	34
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33
28	9.77 405	17	9.86 868	26	0.13 132	9.90 537	10	32
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	9	31
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	10	29
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	9	28
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	10	27
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	9	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	10	24
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	9	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	10	21
40	9.77 609	17	9.87 185	26	0.12 815	9.90 424	9	20
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415	10	19
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	9	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	10	17
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	9	16
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	10	14
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10
51	9.77 795	17	9.87 475	26	0.12 525	9.90 320	9	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	10	8
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	9	7
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	10	6
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	9	5
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	10	4
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	9	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	10	2
59	9.77 930	16	9.87 685	26	0.12 315	9.90 244	9	1
60	9.77 946		9.87 711		0.12 289	9.90 235		0
	L Cos	d	L Cot	cd	L Tan	L Sin	d	′

P P

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	18	17	16
1	1.8	1.7	1.6
2	3.6	3.4	3.2
3	5.4	5.1	4.8
4	7.2	6.8	6.4
5	9.0	8.5	8.0
6	10.8	10.2	9.6
7	12.6	11.9	11.2
8	14.4	13.6	12.8
9	16.2	15.3	14.4

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

P P

53°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	60
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12.026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	17	16
1	1.7	1.6
2	3.4	3.2
3	5.1	4.8
4	6.8	6.4
5	8.5	8.0
6	10.2	9.6
7	11.9	11.2
8	13.6	12.8
9	15.3	14.4

	10	9
1	1.0	0.9
2	2.0	1.8
3	3.0	2.7
4	4.0	3.6
5	5.0	4.5
6	6.0	5.4
7	7.0	6.3
8	8.0	7.2
9	9.0	8.1

P P

38°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	17	16	15
1	1.7	1.6	1.5
2	3.4	3.2	3.0
3	5.1	4.8	4.5
4	6.8	6.4	6.0
5	8.5	8.0	7.5
6	10.2	9.6	9.0
7	11.9	11.2	10.5
8	13.6	12.8	12.0
9	15.3	14.4	13.5

	11	10	9
1	1.1	1.0	0.9
2	2.2	2.0	1.8
3	3.3	3.0	2.7
4	4.4	4.0	3.6
5	5.5	5.0	4.5
6	6.6	6.0	5.4
7	7.7	7.0	6.3
8	8.8	8.0	7.2
9	9.9	9.0	8.1

P P

51°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

'	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.80 867	15	9.92 381	26	0.07 619	9.88 425	10	60
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	30
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	'

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	11	10
1	1.1	1.0
2	2.2	2.0
3	3.3	3.0
4	4.4	4.0
5	5.5	5.0
6	6.6	6.0
7	7.7	7.0
8	8.8	8.0
9	9.9	9.0

P P

41°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	0.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	0.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	30
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	20
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	10
51	9.82 424	15	9.95 215	25	0.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	15	14
1	1.5	1.4
2	3.0	2.8
3	4.5	4.2
4	6.0	5.6
5	7.5	7.0
6	9.0	8.4
7	10.5	9.8
8	12.0	11.2
9	13.5	12.6

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

P P

48°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495	25	0.04 505	9.87 085	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	50
11	9.82 705	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825	25	0.04 175	9.86 936	12	45
16	9.82 775	13	9.95 850	25	0.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	40
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155	25	0.03 845	9.86 786	11	32
29	9.82 955	13	9.96 180	25	0.03 820	9.86 775	12	31
30	9.82 968	14	9.96 205	26	0.03 795	9.86 763	11	30
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	20
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	10
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915	25	0.03 085	9.86 436	11	2
59	9.83 365	13	9.96 940	26	0.03 060	9.86 425	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

P P

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86·306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9 83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9 85 984	12	24
37	9 83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9 85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9 98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7.8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7

	12	11
1	1.2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9

P P

46°

′	L Sin	d	L Tan	c d	L Cot	L Cos	d	
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	60
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	0.01 415	9.85 645	13	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	55
6	9.84 255	14	9.98 635	26	0.01 365	9.85 620	12	54
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	13	52
9	9.84 295	13	9.98 711	26	0.01 289	9.85 583	12	51
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	13	48
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	13	45
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44
17	9.84 398	13	9.98 913	26	0.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	13	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015	25	0.00 985	9.85 436	13	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38
23	9.84 476	13	9.99 065	25	0.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	13	36
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	35
26	9.84 515	13	9.99 141	25	0.00 859	9.85 374	13	34
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	13	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	30
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	13	29
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28
33	9.84 605	13	9.99 318	25	0.00 682	9.85 287	13	27
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	25	0.00 606	9.85 250	13	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225	13	22
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21
40	9.84 694	13	9.99 495	25	0.00 505	9.85 200	13	20
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19
42	9.84 720	13	9.99 545	25	0.00 455	9.85 175	13	18
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17
44	9.84 745	13	9.99 596	25	0.00 404	9.85 150	13	16
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	12	15
46	9.84 771	13	9.99 646	26	0.00 354	9.85 125	13	14
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	13	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	10
51	9.84 835	12	9.99 773	25	0.00 227	9.85 062	13	9
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	13	7
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	13	5
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	13	2
59	9.84 936	13	9.99 975	25	0.00 025	9.84 961	12	1
60	9.84 949		0.00 000		0.00 000	9.84 949		0
	L Cos	d	L Cot	c d	L Tan	L Sin	d	′

P P

	26	25
1	2.6	2.5
2	5.2	5.0
3	7.8	7.5
4	10.4	10.0
5	13.0	12.5
6	15.6	15.0
7	18.2	17.5
8	20.8	20.0
9	23.4	22.5

	14	13	12
1	1.4	1.3	1.2
2	2.8	2.6	2.4
3	4.2	3.9	3.6
4	5.6	5.2	4.8
5	7.0	6.5	6.0
6	8.4	7.8	7.2
7	9.8	9.1	8.4
8	11.2	10.4	9.6
9	12.6	11.7	10.8

P P

TABLE III

LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

From 0° 0′ to 0° 3′, and from 89° 57′ to 90°, for every second.
From 0° to 2°, and from 88° to 90°, for every ten seconds.

Note. — The characteristic of every logarithm in the following table is too large by 10. Therefore, −10 should be written after every logarithm.

L Sin and L Tan — 0° — L Sin and L Tan

″	0′	1′	2′	″	″	0′	1′	2′	″
0	—	6.46 373	6.76 476	60	30	6.16 270	6.63 982	6.86 167	30
1	4.68 557	6.47 090	6.76 836	59	31	6 17 694	6.64 462	6.86 455	29
2	4.98 660	6.47 797	6.77 193	58	32	6 19 072	6.64 936	6.86 742	28
3	5.16 270	6.48 492	6.77 548	57	33	6 20 409	6.65 406	6.87 027	27
4	5.28 763	6.49 175	6.77 900	56	34	6 21 705	6.65 870	6.87 310	26
5	5.38 454	6.49 849	6.78 248	55	35	6 22 964	6.66 330	6.87 591	25
6	5.46 373	6.50 512	6.78 595	54	36	6.24 188	6.66 785	6.87 870	24
7	5.53 067	6.51 165	6.78 938	53	37	6.25 378	6.67 235	6.88 147	23
8	5 58 866	6.51 808	6.79 278	52	38	6 26 536	6.67 680	6.88 423	22
9	5.63 982	6.52 442	6.79 616	51	39	6.27 664	6.68 121	6.88 697	21
10	5.68 557	6.53 067	6.79 952	50	40	6.28 763	6.68 557	6.88 969	20
11	5.72 697	6.53 683	6.80 285	49	41	6.29 836	6.68 990	6.89 240	19
12	5.76 476	6.54 291	6.80 615	48	42	6.30 882	6.69 418	6.89 509	18
13	5.79 952	6.54 890	6.80 943	47	43	6.31 904	6.69 841	6.89 776	17
14	5.83 170	6.55 481	6.81 268	46	44	6.32 903	6.70 261	6.90 042	16
15	5.86 167	6.56 064	6.81 591	45	45	6.33 879	6.70 676	6.90 306	15
16	5.88 969	6.56 639	6.81 911	44	46	6.34 833	6.71 088	6.90 568	14
17	5.91 602	6.57 207	6.82 230	43	47	6 35 767	6.71 496	6.90 829	13
18	5.94 085	6.57 767	6.82 545	42	48	6 36 682	6.71 900	6.91 088	12
19	5.96 433	6.58 320	6.82 859	41	49	6.37 577	6.72 300	6.91 346	11
20	5.98 660	6.58 866	6.83 170	40	50	6.38 454	6.72 697	6.91 602	10
21	6.00 779	6.59 406	6.83 479	39	51	6.39 315	6.73 090	6.91 857	9
22	6.02 800	6.59 939	6.83 786	38	52	6.40 158	6.73 479	6.92 110	8
23	6.04 730	6.60 465	6.84 091	37	53	6.40 985	6 73 865	6.92 362	7
24	6.06 579	6.60 985	6.84 394	36	54	6.41 797	6.74 248	6.92 612	6
25	6.08 351	6.61 499	6.84 694	35	55	6.42 594	6.74 627	6.92 861	5
26	6.10 055	6.62 007	6.84 993	34	56	6.43 376	6.75 003	6.93 109	4
27	6.11 694	6 62 509	6.85 289	33	57	6.44 145	6.75 376	6.93 355	3
28	6 13 273	6.63 006	6.85 584	32	58	6.44 900	6.75 746	6.93 599	2
29	6.14 797	6.63 496	6.85 876	31	59	6.45 643	6.76 112	6.93 843	1
30	6.16 270	6.63 982	6.86 167	30	60	6.46 373	6.76 476	6.94 085	0
″	59′	58′	57′	″	″	59′	58′	57′	″

L Cos and L Cot — 89° — L Cos and L Cot

0°

′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
0 0	—	—	10.00000	0 60	10 0	7.46 373	7.46 373	10.00000	0 50
10	5.68 557	5.68 557	10.00000	50	10	7.47 090	7.47 091	10.00000	50
20	5.98 660	5.98 660	10.00000	40	20	7.47 797	7.47 797	10.00000	40
30	6.16 270	6.16 270	10.00000	30	30	7.48 491	7.48 492	10.00000	30
40	6.28 763	6.28 763	10.00000	20	40	7.49 175	7.49 176	10.00000	20
50	6.38 454	6.38 454	10.00000	10	50	7.49 849	7.49 849	10.00000	10
1 0	6.46 373	6.46 373	10.00000	0 59	11 0	7.50 512	7.50 512	10.00000	0 49
10	6.53 067	6.53 067	10.00000	50	10	7.51 165	7.51 165	10.00000	50
20	6.58 866	6.58 866	10.00000	40	20	7.51 808	7.51 809	10.00000	40
30	6.63 982	6.63 982	10.00000	30	30	7.52 442	7.52 443	10.00000	30
40	6.68 557	6.68 557	10.00000	20	40	7.53 067	7.53 067	10.00000	20
50	6.72 697	6.72 697	10.00000	10	50	7.53 683	7.53 683	10.00000	10
2 0	6.76 476	6.76 476	10.00000	0 58	12 0	7.54 291	7.54 291	10.00000	0 48
10	6.79 952	6.79 952	10.00000	50	10	7.54 890	7.54 890	10.00000	50
20	6.83 170	6.83 170	10.00000	40	20	7.55 481	7.55 481	10.00000	40
30	6.86 167	6.86 167	10.00000	30	30	7.56 064	7.56 064	10.00000	30
40	6.88 969	6.88 969	10.00000	20	40	7.56 639	7.56 639	10.00000	20
50	6.91 602	6.91 602	10.00000	10	50	7.57 206	7.57 207	10.00000	10
3 0	6.94 085	6.94 085	10.00000	0 57	13 0	7.57 767	7.57 767	10.00000	0 47
10	6.96 433	6.96 433	10.00000	50	10	7.58 320	7.58 320	10.00000	50
20	6.98 660	6.98 661	10.00000	40	20	7.58 866	7.58 867	10.00000	40
30	7.00 779	7.00 779	10.00000	30	30	7.59 406	7.59 406	10.00000	30
40	7.02 800	7.02 800	10.00000	20	40	7.59 939	7.59 939	10.00000	20
50	7.04 730	7.04 730	10.00000	10	50	7.60 465	7.60 466	10.00000	10
4 0	7.06 579	7.06 579	10.00000	0 56	14 0	7.60 985	7.60 986	10.00000	0 46
10	7.08 351	7.08 352	10.00000	50	10	7.61 499	7.61 500	10.00000	50
20	7.10 055	7.10 055	10.00000	40	20	7.62 007	7.62 008	10.00000	40
30	7.11 694	7.11 694	10.00000	30	30	7.62 509	7.62 510	10.00000	30
40	7.13 273	7.13 273	10.00000	20	40	7.63 006	7.63 006	10.00000	20
50	7.14 797	7.14 797	10.00000	10	50	7.63 496	7.63 497	10.00000	10
5 0	7.16 270	7.16 270	10.00000	0 55	15 0	7.63 982	7.63 982	10.00000	0 45
10	7.17 694	7.17 694	10.00000	50	10	7.64 461	7.64 462	10.00000	50
20	7.19 072	7.19 073	10.00000	40	20	7.64 936	7.64 937	10.00000	40
30	7.20 409	7.20 409	10.00000	30	30	7.65 406	7.65 406	10.00000	30
40	7.21 705	7.21 705	10.00000	20	40	7.65 870	7.65 871	10.00000	20
50	7.22 964	7.22 964	10.00000	10	50	7.66 330	7.66 330	10.00000	10
6 0	7.24 188	7.24 188	10.00000	0 54	16 0	7.66 784	7.66 785	10.00000	0 44
10	7.25 378	7.25 378	10.00000	50	10	7.67 235	7.67 235	10.00000	50
20	7.26 536	7.26 536	10.00000	40	20	7.67 680	7.67 680	10.00000	40
30	7.27 664	7.27 664	10.00000	30	30	7.68 121	7.68 121	10.00000	30
40	7.28 763	7.28 764	10.00000	20	40	7.68 557	7.68 558	9.99999	20
50	7.29 836	7.29 836	10.00000	10	50	7.68 989	7.68 990	9.99999	10
7 0	7.30 882	7.30 882	10.00000	0 53	17 0	7.69 417	7.69 418	9.99 999	0 43
10	7.31 904	7.31 904	10.00000	50	10	7.69 841	7.69 842	9.99 999	50
20	7.32 903	7.32 903	10.00000	40	20	7.70 261	7.70 261	9.99 999	40
30	7.33 879	7.33 879	10.00000	30	30	7.70 676	7.70 677	9.99 999	30
40	7.34 833	7.34 833	10.00000	20	40	7.71 088	7.71 088	9.99 999	20
50	7.35 767	7.35 767	10.00000	10	50	7 71 496	7 71 496	9.99 999	10
8 0	7.36 682	7.36 682	10.00000	0 52	18 0	7.71 900	7.71 900	9.99 999	0 42
10	7.37 577	7.37 577	10.00000	50	10	7.72 300	7.72 301	9.99 999	50
20	7.38 454	7.38 455	10.00000	40	20	7.72 697	7 72 697	9.99 999	40
30	7.39 314	7.39 315	10.00000	30	30	7.73 090	7.73 090	9.99 999	30
40	7.40 158	7.40 158	10.00000	20	40	7.73 479	7.73 480	9.99 999	20
50	7.40 985	7.40 985	10.00000	10	50	7.73 865	7.73 866	9.99 999	10
9 0	7.41 797	7.41 797	10.00000	0 51	19 0	7.74 248	7.74 248	9.99 999	0 41
10	7.42 594	7.42 594	10.00000	50	10	7.74 627	7.74 628	9.99 999	50
20	7.43 376	7.43 376	10.00000	40	20	7.75 003	7.75 004	9.99 999	40
30	7.44 145	7.44 145	10.00000	30	30	7.75 376	7.75 377	9.99 999	30
40	7.44 900	7.44 900	10.00000	20	40	7.75 745	7.75 746	9.99 999	20
50	7.45 643	7.45 643	10.00000	10	50	7.76 112	7.76 113	9.99 999	10
10 0	7.46 373	7.46 373	10.00000	0 50	20 0	7.76 475	7.76 476	9.99 999	0 40
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

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′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
20 0	7.76 475	7.76 476	9.99 999	0 40	30 0	7.94 084	7.94 086	9.99 998	0 30
10	7.76 836	7.76 837	9.99 999	50	10	7.94 325	7.94 326	9.99 998	50
20	7.77 193	7.77 194	9.99 999	40	20	7.94 564	7.94 566	9.99 998	40
30	7.77 548	7.77 549	9.99 999	30	30	7.94 802	7.94 804	9.99 998	30
40	7.77 899	7.77 900	9.99 999	20	40	7.95 039	7.95 040	9.99 998	20
50	7.78 248	7.78 249	9.99 999	10	50	7.95 274	7.95 276	9.99 998	10
21 0	7.78 594	7.78 595	9.99 999	0 39	31 0	7.95 508	7.95 510	9.99 998	0 29
10	7.78 938	7.78 938	9.99 999	50	10	7.95 741	7.95 743	9.99 998	50
20	7.79 278	7.79 279	9.99 999	40	20	7.95 973	7.95 974	9.99 998	40
30	7.79 616	7.79 617	9.99 999	30	30	7.96 203	7.96 205	9.99 998	30
40	7.79 952	7.79 952	9.99 999	20	40	7.96 432	7.96 434	9.99 998	20
50	7.80 284	7.80 285	9.99 999	10	50	7.96 660	7.96 662	9.99 998	10
22 0	7.80 615	7.80 615	9.99 999	0 38	32 0	7.96 887	7.96 889	9.99 998	0 28
10	7.80 942	7.80 943	9.99 999	50	10	7.97 113	7.97 114	9.99 998	50
20	7.81 268	7.81 269	9.99 999	40	20	7.97 337	7.97 339	9.99 998	40
30	7.81 591	7.81 591	9.99 999	30	30	7.97 560	7.97 562	9.99 998	30
40	7.81 911	7.81 912	9.99 999	20	40	7.97 782	7.97 784	9.99 998	20
50	7.82 229	7.82 230	9.99 999	10	50	7.98 003	7.98 005	9.99 998	10
23 0	7.82 545	7.82 546	9.99 999	0 37	33 0	7.98 223	7.98 225	9.99 998	0 27
10	7.82 859	7.82 860	9.99 999	50	10	7.98 442	7.98 444	9.99 998	50
20	7.83 170	7.83 171	9.99 999	40	20	7.98 660	7.98 662	9.99 998	40
30	7.83 479	7.83 480	9.99 999	30	30	7.98 876	7.98 878	9.99 998	30
40	7.83 786	7.83 787	9.99 999	20	40	7.99 092	7.99 094	9.99 998	20
50	7.84 091	7.84 092	9.99 999	10	50	7.99 306	7.99 308	9.99 998	10
24 0	7.84 393	7.84 394	9.99 999	0 36	34 0	7.99 520	7.99 522	9.99 998	0 26
10	7.84 694	7.84 695	9.99 999	50	10	7.99 732	7.99 734	9.99 998	50
20	7.84 992	7.84 993	9.99 999	40	20	7.99 943	7.99 946	9.99 998	40
30	7.85 289	7.85 290	9.99 999	30	30	8.00 154	8.00 156	9.99 998	30
40	7.85 583	7.85 584	9.99 999	20	40	8.00 363	8.00 365	9.99 998	20
50	7.85 876	7.85 877	9.99 999	10	50	8.00 571	8.00 574	9.99 998	10
25 0	7.86 166	7.86 167	9.99 999	0 35	35 0	8.00 779	8.00 781	9.99 998	0 25
10	7.86 455	7.86 456	9.99 999	50	10	8.00 985	8.00 987	9.99 998	50
20	7.86 741	7.86 743	9.99 999	40	20	8.01 190	8.01 193	9.99 998	40
30	7.87 026	7.87 027	9.99 999	30	30	8.01 395	8.01 397	9.99 998	30
40	7.87 309	7.87 310	9.99 999	20	40	8.01 598	8.01 600	9.99 998	20
50	7.87 590	7.87 591	9.99 999	10	50	8.01 801	8.01 803	9.99 998	10
26 0	7.87 870	7.87 871	9.99 999	0 34	36 0	8.02 002	8.02 004	9.99 998	0 24
10	7.88 147	7.88 148	9.99 999	50	10	8.02 203	8.02 205	9.99 998	50
20	7.88 423	7.88 424	9.99 999	40	20	8.02 402	8.02 405	9.99 998	40
30	7.88 697	7.88 698	9.99 999	30	30	8.02 601	8.02 604	9.99 998	30
40	7.88 969	7.88 970	9.99 999	20	40	8.02 799	8.02 801	9.99 998	20
50	7.89 240	7.89 241	9.99 999	10	50	8.02 996	8.02 998	9.99 998	10
27 0	7.89 509	7.89 510	9.99 999	0 33	37 0	8.03 192	8.03 194	9.99 997	0 23
10	7.89 776	7.89 777	9.99 999	50	10	8.03 387	8.03 390	9.99 997	50
20	7.90 041	7.90 043	9.99 999	40	20	8.03 581	8.03 584	9.99 997	40
30	7.90 305	7.90 307	9.99 999	30	30	8.03 775	8.03 777	9.99 997	30
40	7.90 568	7.90 569	9.99 999	20	40	8.03 967	8.03 970	9.99 997	20
50	7.90 829	7.90 830	9.99 999	10	50	8.04 159	8.04 162	9.99 997	10
28 0	7.91 088	7.91 089	9.99 999	0 32	38 0	8.04 350	8.04 353	9.99 997	0 22
10	7.91 346	7.91 347	9.99 999	50	10	8.04 540	8.04 543	9.99 997	50
20	7.91 602	7.91 603	9.99 999	40	20	8.04 729	8.04 732	9.99 997	40
30	7.91 857	7.91 858	9.99 999	30	30	8.04 918	8.04 921	9.99 997	30
40	7.92 110	7.92 111	9.99 998	20	40	8.05 105	8.05 108	9.99 997	20
50	7.92 362	7.92 363	9.99 998	10	50	8.05 292	8.05 295	9.99 997	10
29 0	7.92 612	7.92 613	9.99 998	0 31	39 0	8.05 478	8.05 481	9.99 997	0 21
10	7.92 861	7.92 862	9.99 998	50	10	8.05 663	8.05 666	9.99 997	50
20	7.93 108	7.93 110	9.99 998	40	20	8.05 848	8.05 851	9.99 997	40
30	7.93 354	7.93 356	9.99 998	30	30	8.06 031	8.06 034	9.99 997	30
40	7.93 599	7.93 601	9.99 998	20	40	8.06 214	8.06 217	9.99 997	20
50	7.93 842	7.93 844	9.99 998	10	50	8.06 396	8.06 399	9.99 997	10
30 0	7.94 084	7.94 086	9.99 998	0 30	40 0	8.06 578	8.06 581	9.99 997	0 20
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

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′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
40 0	8.06 578	8.06 581	9.99 997	0 20	50 0	8.16 268	8.16 273	9.99 995	0 10
10	8.06 758	8.06 761	9.99 997	50	10	8.16 413	8.16 417	9.99 995	50
20	8.06 938	8.06 941	9.99 997	40	20	8.16 557	8.16 561	9.99 995	40
30	8.07 117	8.07 120	9.99 997	30	30	8.16 700	8.16 705	9.99 995	30
40	8.07 295	8.07 299	9.99 997	20	40	8.16 843	8.16 848	9.99 995	20
50	8.07 473	8.07 476	9.99 997	10	50	8.16 986	8.16 991	9.99 995	10
41 0	8.07 650	8.07 653	9.99 997	0 19	51 0	8.17 128	8.17 133	9.99 995	0 9
10	8.07 826	8.07 829	9.99 997	50	10	8.17 270	8.17 275	9.99 995	50
20	8.08 002	8.08 005	9.99 997	40	20	8.17 411	8.17 416	9.99 995	40
30	8.08 176	8.08 180	9.99 997	30	30	8.17 552	8.17 557	9.99 995	30
40	8.08 350	8.08 354	9.99 997	20	40	8.17 692	8.17 697	9.99 995	20
50	8.08 524	8.08 527	9.99 997	10	50	8.17 832	8.17 837	9.99 995	10
42 0	8.08 696	8.08 700	9.99 997	0 18	52 0	8.17 971	8.17 976	9.99 995	0 8
10	8.08 868	8.08 872	9.99 997	50	10	8.18 110	8.18 115	9.99 995	50
20	8.09 040	8.09 043	9.99 997	40	20	8.18 249	8.18 254	9.99 995	40
30	8.09 210	8.09 214	9.99 997	30	30	8.18 387	8.18 392	9.99 995	30
40	8.09 380	8.09 384	9.99 997	20	40	8.18 524	8.18 530	9.99 995	20
50	8.09 550	8.09 553	9.99 997	10	50	8.18 662	8.18 667	9.99 995	10
43 0	8.09 718	8.09 722	9.99 997	0 17	53 0	8.18 798	8.18 804	9.99 995	0 7
10	8.09 886	8.09 890	9.99 997	50	10	8.18 935	8.18 940	9.99 995	50
20	8.10 054	8.10 057	9.99 997	40	20	8.19 071	8.19 076	9.99 995	40
30	8.10 220	8.10 224	9.99 997	30	30	8.19 206	8.19 212	9.99 995	30
40	8.10 386	8.10 390	9.99 997	20	40	8.19 341	8.19 347	99.9 995	20
50	8.10 552	8.10 555	9.99 996	10	50	8.19 476	8.19 481	9.99 995	10
44 0	8.10 717	8.10 720	9.99 996	0 16	54 0	8.19 610	8.19 616	9.99 905	0 6
10	8.10 881	8.10 884	9.99 996	50	10	8.19 744	8.19 749	9.99 995	50
20	8.11 044	8.11 048	9.99 996	40	20	8.19 877	8.19 883	9.99 995	40
30	8.11 207	8.11 211	9.99 996	30	30	8.20 010	8.20 016	9.99 995	30
40	8.11 370	8.11 373	9.99 996	20	40	8.20 143	8.20 149	9.99 995	20
50	8.11 531	8.11 535	9.99 996	10	50	8.20 275	8.20 281	9.99 994	10
45 0	8.11 693	8.11 696	9.99 996	0 15	55 0	8.20 407	8.20 413	9.99 994	0 5
10	8.11 853	8.11 857	9.99 996	50	10	8.20 538	8.20 544	9.99 994	50
20	8.12 013	8.12 017	9.99 996	40	20	8.20 669	8.20 675	9.99 994	40
30	8.12 172	8.12 176	9.99 996	30	30	8.20 800	8.20 806	9.99 994	30
40	8.12 331	8.12 335	9.99 996	20	40	8.20 930	8.20 936	9.99 994	20
50	8.12 489	8.12 493	9.99 996	10	50	8.21 060	8.21 066	9.99 994	10
46 0	8.12 647	8.12 651	9.99 996	0 14	56 0	8.21 189	8.21 195	9.99 994	0 4
10	8.12 804	8.12 808	9.99 996	50	10	8.21 319	8.21 324	9.99 994	50
20	8.12 961	8.12 965	9.99 996	40	20	8.21 447	8.21 453	9.99 994	40
30	8.13 117	8.13 121	9.99 996	30	30	8.21 576	8.21 581	9.99 994	30
40	8.13 272	8.13 276	9.99 996	20	40	8.21 703	8.21 709	9.99 994	20
50	8.13 427	8.13 431	9.99 996	10	50	8.21 831	8.21 837	9.99 994	10
47 0	8.13 581	8.13 585	9.99 996	0 13	57 0	8.21 958	8.21 964	9.99 994	0 3
10	8.13 735	8.13 739	9.99 996	50	10	8.22 085	8.22 091	9.99 994	50
20	8.13 888	8.13 892	9.99 996	40	20	8.22 211	8.22 217	9.99 994	40
30	8.14 041	8.14 045	9.99 996	30	30	8.22 337	8.22 343	9.99 994	30
40	8.14 193	8.14 197	9.99 996	20	40	8.22 463	8.22 469	9.99 994	20
50	8.14 344	8.14 348	9.99 996	10	50	8.22 588	8.22 595	9.99 994	10
48 0	8.14 495	8.14 500	9.99 996	0 12	58 0	8.22 713	8.22 720	9.99 994	0 2
10	8.14 646	8.14 650	9.99 996	50	10	8.22 838	8.22 844	9.99 994	50
20	8.14 796	8.14 800	9.99 996	40	20	8.22 962	8.22 968	9.99 994	40
30	8.14 945	8.14 950	9.99 996	30	30	8.23 086	8.23 092	9.99 994	30
40	8.15 094	8.15 099	9.99 996	20	40	8.23 210	8.23 216	9.99 994	20
50	8.15 243	8.15 247	9.99 996	10	50	8.23 333	8.23 339	9.99 994	10
49 0	8.15 391	8.15 395	9.99 996	0 11	59 0	8.23 456	8.23 462	9.99 994	0 1
10	8.15 538	8.15 543	9.99 996	50	10	8.23 578	8.23 585	9.99 994	50
20	8.15 685	8.15 690	9.99 996	40	20	8.23 700	8.23 707	9.99 904	40
30	8.15 832	8.15 836	9.99 996	30	30	8.23 822	8.23 829	9.99 993	30
40	8.15 978	8.15 982	9.99 995	20	40	8.23 944	8.23 950	9.99 993	20
50	8.16 123	8.16 128	9.99 995	10	50	8.24 065	8.24 071	9.99 993	10
50 0	8.16 268	8.16 273	9.99 995	0 10	60 0	8.24 186	8.24 192	9.99 993	0 0
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

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′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
0 0	8.24 186	8.24 192	9.99 993	0 60	10 0	8.30 879	8.30 888	9.99 991	0 50
10	8.24 306	8.24 313	9.99 993	50	10	8.30 983	8.30 992	9.99 991	50
20	8.24 426	8.24 433	9.99 993	40	20	8.31 086	8.31 095	9.99 991	40
30	8.24 546	8.24 553	9.99 993	30	30	8.31 188	8.31 198	9.99 991	30
40	8.24 665	8.24 672	9.99 993	20	40	8.31 291	8.31 300	9.99 991	20
50	8.24 785	8.24 791	9.99 993	10	50	8.31 393	8.31 403	9.99 991	10
1 0	8.24 903	8.24 910	9.99 993	0 59	11 0	8.31 495	8.31 505	9.99 991	0 49
10	8.25 022	8.25 029	9.99 993	50	10	8.31 597	8.31 606	9.99 991	50
20	8.25 140	8.25 147	9.99 993	40	20	8.31 699	8.31 708	9.99 991	40
30	8.25 258	8.25 265	9.99 993	30	30	8.31 800	8.31 809	9.99 991	30
40	8.25 375	8.25 382	9.99 993	20	40	8.31 901	8.31 911	9.99 991	20
50	8.25 493	8.25 500	9.99 993	10	50	8.32 002	8.32 012	9.99 991	10
2 0	8.25 609	8.25 616	9.99 993	0 58	12 0	8.32 103	8.32 112	9.99 990	0 48
10	8.25 726	8.25 733	9.99 993	50	10	8.32 203	8.32 213	9.99 990	50
20	8.25 842	8.25 849	9.99 993	40	20	8.32 303	8.32 313	9.99 990	40
30	8.25 958	8.25 965	9.99 993	30	30	8.32 403	8.32 413	9.99 990	30
40	8.26 074	8.26 081	9.99 993	20	40	8.32 503	8.32 513	9.99 990	20
50	8.26 189	8.26 196	9.99 993	10	50	8.32 602	8.32 612	9.99 990	10
3 0	8.26 304	8.26 312	9.99 993	0 57	13 0	8.32 702	8.32 711	9.99 990	0 47
10	8.26 419	8.26 426	9.99 993	50	10	8.32 801	8.32 811	9.99 990	50
20	8.26 533	8.26 541	9.99 993	40	20	8.32 899	8.32 909	9.99 990	40
30	8.26 648	8.26 655	9.99 993	30	30	8.32 998	8.33 008	9.99 990	30
40	8.26 761	8.26 769	9.99 993	20	40	8.33 096	8.33 106	9.99 990	20
50	8.26 875	8.26 882	9.99 993	10	50	8.33 195	8.33 205	9.99 990	10
4 0	8.26 988	8.26 996	9.99 992	0 56	14 0	8.33 292	8.33 302	9.99 990	0 46
10	8.27 101	8.27 109	9.99 992	50	10	8.33 390	8.33 400	9.99 990	50
20	8.27 214	8.27 221	9.99 992	40	20	8.33 488	8.33 498	9.99 990	40
30	8.27 326	8.27 334	9.99 992	30	30	8.33 585	8.33 595	9.99 990	30
40	8.27 438	8.27 446	9.99 992	20	40	8.33 682	8.33 692	9.99 990	20
50	8.27 550	8.27 558	9.99 992	10	50	8.33 779	8.33 789	9.99 990	10
5 0	8.27 661	8.27 669	9.99 992	0 55	15 0	8.33 875	8.33 886	9.99 990	0 45
10	8.27 773	8.27 780	9.99 992	50	10	8.33 972	8.33 982	9.99 990	50
20	8.27 883	8.27 891	9.99 992	40	20	8.34 068	8.34 078	9.99 990	40
30	8.27 994	8.28 002	9.99 992	30	30	8.34 164	8.34 174	9.99 990	30
40	8.28 104	8.28 112	9.99 992	20	40	8.34 260	8.34 270	9.99 989	20
50	8.28 215	8.28 223	9.99 992	10	50	8.34 355	8.34 366	9.99 989	10
6 0	8.28 324	8.28 332	9.99 992	0 54	16 0	8.34 450	8.34 461	9.99 989	0 44
10	8.28 434	8.28 442	9.99 992	50	10	8.34 546	8.34 556	9.99 989	50
20	8.28 543	8.28 551	9.99 992	40	20	8.34 640	8.34 651	9.99 989	40
30	8.28 652	8.28 660	9.99 992	30	30	8.34 735	8.34 746	9.99 989	30
40	8.28 761	8.28 769	9.99 992	20	40	8.34 830	8.34 840	9.99 989	20
50	8.28 869	8.28 877	9.99 992	10	50	8.34 924	8.34 935	9.99 989	10
7 0	8.28 977	8.28 986	9.99 992	0 53	17 0	8.35 018	8.35 029	9.99 989	0 43
10	8.29 085	8.29 094	9.99 992	50	10	8.35 112	8.35 123	9.99 989	50
20	8.29 193	8.29 201	9.99 992	40	20	8.35 206	8.35 217	9.99 989	40
30	8.29 300	8.29 309	9.99 992	30	30	8.35 299	8.35 310	9.99 989	30
40	8.29 407	8.29 416	9.99 992	20	40	8.35 392	8.35 403	9.99 989	20
50	8.29 514	8.29 523	9.99 992	10	50	8.35 485	8.35 497	9.99 989	10
8 0	8.29 621	8.29 629	9.99 992	0 52	18 0	8.35 578	8.35 590	9.99 989	0 42
10	8.29 727	8.29 736	9.99 991	50	10	8.35 671	8.35 682	9.99 989	50
20	8.29 833	8.29 842	9.99 991	40	20	8.35 764	8.35 775	9.99 989	40
30	8.29 939	8.29 947	9.99 991	30	30	8.35 856	8.35 867	9.99 989	30
40	8.30 044	8.30 053	9.99 991	20	40	8.35 948	8.35 959	9.99 989	20
50	8.30 150	8.30 158	9.99 991	10	50	8.36 040	8.36 051	9.99 989	10
9 0	8.30 255	8.30 263	9.99 991	0 51	19 0	8.36 131	8.36 143	9.99 989	0 41
10	8.30 359	8.30 368	9.99 991	50	10	8.36 223	8.36 235	9.99 988	50
20	8.30 464	8.30 473	9.99 991	40	20	8.36 314	8.36 326	9.99 988	40
30	8.30 568	8.30 577	9.99 991	30	30	8.36 405	8.36 417	9.99 988	30
40	8.30 672	8.30 681	9.99 991	20	40	8.36 496	8.36 508	9.99 988	20
50	8.30 776	8.30 785	9.99 991	10	50	8.36 587	8.36 599	9.99 988	10
10 0	8.30 879	8.30 888	9.99 991	0 50	20 0	8.36 678	8.36 689	9.99 988	0 40
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
20 0	8.36 678	8.36 689	9.99 988	0 **40**	**30** 0	8.41 792	8.41 807	9.99 985	0 **30**
10	8.36 768	8.36 780	9.99 988	50	10	8.41 872	8.41 887	9.99 985	50
20	8.36 858	8.36 870	9.99 988	40	20	8.41 952	8.41 967	9.99 985	40
30	8.36 948	8.36 960	9.99 988	30	30	8.42 032	8.42 048	9.99 985	30
40	8.37 038	8.37 050	9.99 988	20	40	8.42 112	8.42 127	9.99 985	20
50	8.37 128	8.37 140	9.99 988	10	50	8.42 192	8.42 207	9.99 985	10
21 0	8.37 217	8.37 229	9.99 988	0 **39**	**31** 0	8.42 272	8.42 287	9.99 985	0 **29**
10	8.37 306	8.37 318	9.99 988	50	10	8.42 351	8.42 366	9.99 985	50
20	8.37 395	8.37 408	9.99 988	40	20	8.42 430	8.42 446	9.99 985	40
30	8.37 484	8.37 497	9.99 988	30	30	8.42 510	8.42 525	9.99 985	30
40	8.37 573	8.37 585	9.99 988	20	40	8.42 589	8.42 406	9.99 985	20
50	8.37 662	8.37 674	9.99 988	10	50	8.42 667	8.42 683	9.99 985	10
22 0	8.37 750	8.37 762	9.99 988	0 **38**	**32** 0	8.42 746	8.42 762	9.99 984	0 **28**
10	8.37 838	8.37 850	9.99 988	50	10	8.42 825	8.42 840	9.99 984	50
20	8.37 926	8.37 938	9.99 988	40	20	8.42 903	8.42 919	9.99 984	40
30	8.38 014	8.38 026	9.99 987	30	30	8.42 982	8.42 997	9.99 984	30
40	8.38 101	8.38 114	9.99 987	20	40	8.43 060	8.43 075	9.99 984	20
50	8.38 189	8.38 202	9.99 987	10	50	8.43 138	8.43 154	9.99 984	10
23 0	8.38 276	8.38 289	9.99 987	0 **37**	**33** 0	8.43 216	8.43 232	9.99 984	0 **27**
10	8.38 363	8.38 376	9.99 987	50	10	8.43 293	8.43 309	9.99 984	50
20	8.38 450	8.38 463	9.99 987	40	20	8.43 371	8.43 387	9.99 984	40
30	8.38 537	8.38 550	9.99 987	30	30	8.43 448	8.43 464	9.99 984	30
40	8.38 624	8.38 636	9.99 987	20	40	8.43 526	8.43 542	9.99 984	20
50	8.38 710	8.38 723	9.99 987	10	50	8.43 603	8.43 619	9.99 984	10
24 0	8.38 796	8.38 809	9.99 987	0 **36**	**34** 0	8.43 680	8.43 696	9.99 984	0 **26**
10	8.38 882	8.38 895	9.99 987	50	10	8.43 757	8.43 773	9.99 984	50
20	8.38 968	8.38 981	9.99 987	40	20	8.43 834	8.43 850	9.99 984	40
30	8.39 054	8.39 067	9.99 987	30	30	8.43 910	8.43 927	9.99 984	30
40	8.39 139	8.39 153	9.99 987	20	40	8.43 987	8.44 003	9.99 984	20
50	8.39 225	8.39 238	9.99 987	10	50	8.44 063	8.44 080	9.99 983	10
25 0	8.39 310	8.39 323	9.99 987	0 **35**	**35** 0	8.44 139	8.44 156	9.99 983	0 **25**
10	8.39 395	8.39 408	9.99 987	50	10	8.44 216	8.44 232	9.99 983	50
20	8.39 480	8.39 493	9.99 987	40	20	8.44 292	8.44 308	9.99 983	40
30	8.39 565	8.39 587	9.99 987	30	30	8.44 367	8.44 384	9.99 983	30
40	8.39 649	8.39 663	9.99 987	20	40	8.44 443	8.44 460	9.99 983	20
50	8.39 734	8.39 747	9.99 986	10	50	8.44 519	8.44 536	9.99 983	10
26 0	8.39 818	8.39 832	9.99 986	0 **34**	**36** 0	8.44 594	8.44 611	9.99 983	0 **24**
10	8.39 902	8.39 916	9.99 986	50	10	8.44 669	8.44 686	9.99 983	50
20	8.39 986	8.40 000	9.99 986	40	20	8.44 745	8.44 762	9.99 983	40
30	8.40 070	8.40 083	9.99 986	30	30	8.44 820	8.44 837	9.99 983	30
40	8.40 153	8.40 167	9.99 986	20	40	8.44 895	8.44 912	9.99 983	20
50	8.40 237	8.40 251	9.99 986	10	50	8.44 969	8.44 987	9.99 983	10
27 0	8.40 320	8.40 334	9.99 986	0 **33**	**37** 0	8.45 044	8.45 061	9.99 983	0 **23**
10	8.40 403	8.40 417	9.99 986	50	10	8.45 119	8.45 136	9.99 983	50
20	8.40 486	8.40 500	9.99 986	40	20	8.45 193	8.45 210	9.99 983	40
30	8.40 569	8.40 583	9.99 986	30	30	8.45 267	8.45 285	9.99 983	30
40	8.40 651	8.40 665	9.99 986	20	40	8.45 341	8.45 359	9.99 982	20
50	8.40 734	8.40 748	9.99 986	10	50	8.45 415	8.45 433	9.99 982	10
28 0	8.40 816	8.40 830	9.99 986	0 **32**	**38** 0	8.45 489	8.45 507	9.99 982	0 **22**
10	8.40 898	8.40 913	9.99 986	50	10	8.45 563	8.45 581	9.99 982	50
20	8.40 980	8.40 995	9.99 986	40	20	8.45 637	8.45 655	9.99 982	40
30	8.41 062	8.41 077	9.99 986	30	30	8.45 710	8.45 728	9.99 982	30
40	8.41 144	8.41 158	9.99 986	20	40	8.45 784	8.45 802	9.99 982	20
50	8.41 225	8.41 240	9.99 986	10	50	8.45 857	8.45 875	9.99 982	10
29 0	8.41 307	8.41 321	9.99 985	0 **31**	**39** 0	8.45 930	8.45 948	9.99 982	0 **21**
10	8.41 388	8.41 403	9.99 985	50	10	8.46 003	8.46 021	9.99 982	50
20	8.41 469	8.41 484	9.99 985	40	20	8.46 076	8.46 094	9.99 982	40
30	8.41 550	8.41 565	9.99 985	30	30	8.46 149	8.46 167	9.99 982	30
40	8.41 631	8.41 646	9.99 985	20	40	8.46 222	8.46 240	9.99 982	20
50	8.41 711	8.41 726	9.99 985	10	50	8.46 294	8.46 312	9.99 982	10
30 0	8.41 792	8.41 807	9.99 985	0 **30**	**40** 0	8.46 366	8.46 385	9.99 982	0 **20**
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

1°

′ ″	L Sin	L Tan	L Cos	″ ′	′ ″	L Sin	L Tan	L Cos	″ ′
40 0	8.46 366	8.46 385	9.99 982	0 20	50 0	8.50 504	8.50 527	9.99 978	0 10
10	8.46 439	8.46 457	9.99 982	50	10	8.50 570	8.50 593	9.99 978	50
20	8.46 511	8.46 529	9.99 982	40	20	8.50 636	8.50 658	9.99 978	40
30	8.46 583	8.46 602	9.99 981	30	30	8.50 701	8.50 724	9.99 978	30
40	8.46 655	8.46 674	9.99 981	20	40	8.50 767	8.50 789	9.99 977	20
50	8.46 727	8.46 745	9.99 981	10	50	8.50 832	8.50 855	9.99 977	10
41 0	8.46 799	8.46 817	9.99 981	0 19	51 0	8.50 897	8.50 920	9.99 977	0 9
10	8.46 870	8.46 889	9.99 981	50	10	8.50 963	8.50 985	9.99 977	50
20	8.46 942	8.46 960	9.99 981	40	20	8.51 028	8.51 050	9.99 977	40
30	8.47 013	8.47 032	9.99 981	30	30	8.51 092	8.51 015	9.99 977	30
40	8.47 084	8.47 103	9.99 981	20	40	8.51 157	8.51 180	9.99 977	20
50	8.47 155	8.47 174	9.99 981	10	50	8.51 222	8.51 245	9.99 977	10
42 0	8.47 226	8.47 245	9.99 981	0 18	52 0	8.51 287	8.51 310	9.99 977	0 8
10	8.47 297	8.47 316	9.99 981	50	10	8.51 351	8.51 374	9.99 977	50
20	8.47 368	8.47 387	9.99 981	40	20	8.51 416	8.51 439	9.99 977	40
30	8.47 439	8.47 458	9.99 981	30	30	8.51 480	8.51 503	9.99 977	30
40	8.47 509	8.47 528	9.99 981	20	40	8.51 544	8.51 568	9.99 977	20
50	8.47 580	8.47 599	9.99 981	10	50	8.51 609	8.51 632	9.99 977	10
43 0	8.47 650	8.47 669	9.99 981	0 17	53 0	8.51 673	8.51 696	9.99 977	0 7
10	8.47 720	8.47 740	9.99 980	50	10	8.51 737	8.51 760	9.99 976	50
20	8.47 790	8.47 810	9.99 980	40	20	8.51 801	8.51 824	9.99 976	40
30	8.47 860	8.47 880	9.99 980	30	30	8.51 864	8.51 888	9.99 976	30
40	8.47 930	8.47 950	9.99 980	20	40	8.51 928	8.51 952	9.99 976	20
50	8.48 000	8.48 020	9.99 980	10	50	8.51 992	8.52 015	9.99 976	10
44 0	8.48 096	8.48 090	9.99 980	0 16	54 0	8.52 055	8.52 079	9.99 976	0 6
10	8.48 139	8.48 159	9.99 980	50	10	8.52 119	8.52 143	9.99 976	50
20	8.48 208	8.48 228	9.99 980	40	20	8.52 182	8.52 206	9.99 976	40
30	8.48 278	8.48 298	9.99 980	30	30	8.52 245	8.52 269	9.99 976	30
40	8.48 347	8.48 367	9.99 980	20	40	8.52 308	8.52 332	9.99 976	20
50	8.48 416	8.48 436	9.99 980	10	50	8.52 371	8.52 396	9.99 976	10
45 0	8.48 485	8.48 505	9.99 980	0 15	55 0	8.52 434	8.52 459	9.99 976	0 5
10	8.48 554	8.48 574	9.99 980	50	10	8.52 497	8.52 522	9.99 976	50
20	8.48 622	8.48 643	9.99 980	40	20	8.52 560	8.52 584	9.99 976	40
30	8.48 691	8.48 711	9.99 980	30	30	8.52 623	8.52 647	9.99 975	30
40	8.48 760	8.48 780	9.99 979	20	40	8.52 685	8.52 710	9.99 975	20
50	8.48 828	8.48 849	9.99 979	10	50	8.52 748	8.52 772	9.99 975	10
46 0	8.48 896	8.48 917	9.99 979	0 14	56 0	8.52 810	8.52 835	9.99 975	0 4
10	8.48 965	8.48 985	9.99 979	50	10	8.52 872	8.52 897	9.99 975	50
20	8.49 033	8.49 053	9.99 979	40	20	8.52 935	8.52 960	9.99 975	40
30	8.49 101	8.49 121	9.99 979	30	30	8.52 997	8.53 022	9.99 975	30
40	8.49 169	8.49 189	9.99 979	20	40	8.53 059	8.53 084	9.99 975	20
50	8.49 236	8.49 257	9.99 979	10	50	8.53 121	8.53 146	9.99 975	10
47 0	8.49 304	8.49 325	9.99 979	0 13	57 0	8.53 183	8.53 208	9.99 975	0 3
10	8.49 372	8.49 393	9.99 979	50	10	8.53 245	8.53 270	9.99 975	50
20	8.49 439	8.49 460	9.99 979	40	20	8.53 306	8.53 332	9.99 975	40
30	8.49 506	8.49 528	9.99 979	30	30	8.53 368	8.53 393	9.99 975	30
40	8.49 574	8.49 595	9.99 979	20	40	8.53 429	8.53 455	9.99 975	20
50	8.49 641	8.49 662	9.99 979	10	50	8.53 491	8.53 516	9.99 974	10
48 0	8.49 708	8.49 729	9.99 979	0 12	58 0	8.53 552	8.53 578	9.99 974	0 2
10	8.49 775	8.49 796	9.99 979	50	10	8.53 614	8.53 639	9.99 974	50
20	8.49 842	8.49 863	9.99 978	40	20	8.53 675	8.53 700	9.99 974	40
30	8.49 908	8.49 930	9.99 978	30	30	8.53 736	8.53 762	9.99 974	30
40	8.49 975	8.49 997	9.99 978	20	40	8.53 797	8.53 823	9.99 974	20
50	8.50 042	8.50 063	9.99 978	10	50	8.53 858	8.53 884	9.99 974	10
49 0	8.50 108	8.50 130	9.99 978	0 11	59 0	8.53 919	8.53 945	9.99 974	0 1
10	8.50 174	8.50 196	9.99 978	50	10	8.53 979	8.54 005	9.99 974	50
20	8.50 241	8.50 263	9.99 978	40	20	8.54 040	8.54 066	9.99 974	40
30	8.50 307	8.50 329	9.99 978	30	30	8.54 101	8.54 127	9.99 974	30
40	8.50 373	8.50 395	9.99 978	20	40	8.54 161	8.54 187	9.99 974	20
50	8.50 439	8.50 461	9.99 978	10	50	8.54 222	8.54 248	9.99 974	10
50 0	8.50 504	8.50 527	9.99 978	0 10	60 0	8.54 282	8.54 308	9.99 974	0 0
′ ″	L Cos	L Cot	L Sin	″ ′	′ ″	L Cos	L Cot	L Sin	″ ′

88°

TABLE IV

THE AUXILIARIES S' AND T'

1 If θ is an angle between 0° and 2° and θ' represents the number of minutes in the angle θ, the following formulae involving the quantities S' and T' are convenient.

$$\log \sin \theta = \log \theta' + S', \qquad \log \theta' = \log \sin \theta - S',$$
$$\log \tan \theta = \log \theta' + T', \qquad \log \theta' = \log \tan \theta - T',$$
$$\log \cot \theta = \text{colog} \tan \theta, \qquad \log \theta' = \text{colog} \cot \theta - T'$$

2. If θ is an angle between 88° and 90° and $(90° - \theta)'$ represents the number of minutes in the angle $90° - \theta$, we have

$$\log \cos \theta = \log (90° - \theta)' + S', \qquad \log (90° - \theta)' = \log \cos \theta - S',$$
$$\log \cot \theta = \log (90° - \theta)' + T', \qquad \log (90° - \theta)' = \log \cot \theta - T',$$
$$\log \tan \theta = \text{colog} \cot \theta, \qquad \log (90° - \theta)' = \text{colog} \tan \theta - T'.$$

VALUES OF S' AND T'

θ'	S'	log sin θ
0		—
	6.46373	
14		7.60985
	6.46372	
43		8.09718
	6.46371	
59		8.23456
	6.46370	
72		8.32103
	6.46369	
82		8.37750
	6.46368	
92		8.42746
	6.46367	
100		8.46366
	6.46366	
108		8.49708
	6.46365	
116		8.52810
	6.46364	
122		8.54999
	6.46363	

θ	T	log tan θ	θ'	T'	log tan θ
0		—	90		8.41807
	6.46373			6.46383	
27		7.89510	95		8.44156
	6.46374			6.46384	
40		8.06581	99		8.45948
	6.46375			6.46385	
49		8.15395	103		8.47669
	6.46376			6.46386	
57		8.21964	107		8.49325
	6.46377			6.46387	
64		8.26996	111		8.50920
	6.46378			6.46388	
70		8.30888	114		8.52079
	6.46379			6.46389	
75		8.33886	118		8.53578
	6.46380			6.46390	
81		8.37229	121		8.54669
	6.46381			6.46391	
86		8.39832			
	6.46382				
90		8.41807			

TABLE V

FOUR-PLACE VALUES

OF THE

NATURAL TRIGONOMETRIC FUNCTIONS

FOR EVERY TENTH OF A DEGREE FROM 0° TO 90°

0° to 3°

Angle	Sin	d	Tan	d	Cot	d	Cos	d	
0°.0	0.0000	17	0.0000	17	∞		1.0000	0	90°.0
1	0.0017	18	0.0017	18	572.9572		1.0000	0	9
2	0.0035	17	0.0035	17	286.4777		1.0000	0	8
3	0.0052	18	0.0052	18	190.9842		1.0000	0	7
4	0.0070	17	0.0070	17	143.2371		1.0000	0	6
5	0.0087	18	0.0087	18	114.5887		1.0000	1	5
6	0.0105	17	0.0105	17	95.4895		0.9999	0	4
7	0.0122	18	0.0122	18	81.8470		0.9999	0	3
8	0.0140	17	0.0140	17	71.6151		0.9999	0	2
9	0.0157	18	0.0157	18	63.6567		0.9999	1	1
1°.0	0.0175	17	0.0175	17	57.2900		0.9998	0	89°.0
1	0.0192	17	0.0192	17	52.0807		0.9998	0	9
2	0.0209	18	0.0209	18	47.7395		0.9998	1	8
3	0.0227	17	0.0227	17	44.0661		0.9997	0	7
4	0.0244	18	0.0244	18	40.9174		0.9997	0	6
5	0.0262	17	0.0262	17	38.1885		0.9997	1	5
6	0.0279	18	0.0279	18	35.8006		0.9996	0	4
7	0.0297	17	0.0297	17	33.6935		0.9996	1	3
8	0.0314	18	0.0314	18	31.8205		0.9995	0	2
9	0.0332	17	0.0332	17	30.1446		0.9995	1	1
2°.0	0.0349	17	0.0349	18	28.6363		0.9994	1	88°.0
1	0.0366	18	0.0367	17	27.2715		0.9993	0	9
2	0.0384	17	0.0384	18	26.0307		0.9993	1	8
3	0.0401	18	0.0402	17	24.8978		0.9992	1	7
4	0.0419	17	0.0419	18	23.8593	9555	0.9991	1	6
5	0.0436	18	0.0437	17	22.9038	8821	0.9990	0	5
6	0.0454	17	0.0454	18	22.0217	8168	0.9990	1	4
7	0.0471	17	0.0472	17	21.2049	7584	0.9989	1	3
8	0.0488	18	0.0489	18	20.4465	7062	0.9988	1	2
9	0.0506	17	0.0507	17	19.7403	6592	0.9987	1	1
3°.0	0.0523		0.0524		19.0811		0.9986		87°.0
	Cos	d	Cot	d	Tan	d	Sin	d	Angle

P P

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

87° to 90°

Angle	Sin	d	Tan	d	Cot	d	Cos	d	
3°.0	0.0523	18	0.0524	18	19.0811	6166	0.9986	1	87°.0
1	0.0541	17	0.0542	17	18.4645	5782	0.9985	1	9
2	0.0558	18	0.0559	18	17.8863	5431	0.9984	1	8
3	0.0576	17	0.0577	17	17.3432	5113	0.9983	1	7
4	0.0593	17	0.0594	18	16.8319	4820	0.9982	1	6
5	0.0610	18	0.0612	17	16.3499	4554	0.9981	1	5
6	0.0628	17	0.0629	18	15.8945	4307	0.9980	1	4
7	0.0645	18	0.0647	17	15.4638	4081	0.9979	1	3
8	0.0663	17	0.0664	18	15.0557	3872	0.9978	1	2
9	0.0680	18	0.0682	17	14.6685	3678	0.9977	1	1
4°.0	0.0698	17	0.0699	18	14.3007	3500	0.9976	2	86°.0
1	0.0715	17	0.0717	17	13.9507	3333	0.9974	1	9
2	0.0732	18	0.0734	18	13.6174	3179	0.9973	1	8
3	0.0750	17	0.0752	17	13.2995	3033	0.9972	1	7
4	0.0767	18	0.0769	18	12.9962	2900	0.9971	2	6
5	0.0785	17	0.0787	18	12.7062	2774	0.9969	1	5
6	0.0802	17	0.0805	17	12.4288	2666	0.9968	2	4
7	0.0819	18	0.0822	18	12.1622	2535	0.9966	1	3
8	0.0837	17	0.0840	17	11.9087	2442	0.9965	2	2
9	0.0854	18	0.0857	18	11.6645	2344	0.9963	1	1
5°.0	0.0872	17	0.0875	17	11.4301	2253	0.9962	2	85°.0
1	0.0889	17	0.0892	18	11.2048	2166	0.9960	1	9
2	0.0906	18	0.0910	18	10.9882	2085	0.9959	2	8
3	0.0924	17	0.0928	17	10.7797	2008	0.9957	2	7
4	0.0941	17	0.0945	18	10.5789	1935	0.9956	2	6
5	0.0958	18	0.0963	18	10.3854	1866	0.9954	2	5
6	0.0976	17	0.0981	17	10.1988	1801	0.9952	1	4
7	0.0993	18	0.0998	18	10.0187	1739	0.9951	2	3
8	0.1011	17	0.1016	17	9.8448	1680	0.9949	2	2
9	0.1028	17	0.1033	18	9.6768	1624	0.9947	2	1
6°.0	0.1045	18	0.1051	18	9.5144	1572	0.9945	2	84°.0
1	0.1063	17	0.1069	17	9.3572	1520	0.9943	1	9
2	0.1080	17	0.1086	18	9.2052	1473	0.9942	2	8
3	0.1097	18	0.1104	18	9.0579	1427	0.9940	2	7
4	0.1115	17	0.1122	17	8.9152	1383	0.9938	2	6
5	0.1132	17	0.1139	18	8.7769	1342	0.9936	2	5
6	0.1149	18	0.1157	18	8.6427	1301	0.9934	2	4
7	0.1167	17	0.1175	17	8.5126	1263	0.9932	2	3
8	0.1184	17	0.1192	18	8.3863	1227	0.9930	2	2
9	0.1201	18	0.1210	18	8.2636	1193	0.9928	3	1
7°.0	0.1219	17	0.1228	18	8.1443	1158	0.9925	2	83°.0
1	0.1236	17	0.1246	17	8.0285	1127	0.9923	2	9
2	0.1253	18	0.1263	18	7.9158	1096	0.9921	2	8
3	0.1271	17	0.1281	18	7.8062	1066	0.9919	2	7
4	0.1288	17	0.1299	18	7.6996	1038	0.9917	3	6
5	0.1305	18	0.1317	17	7.5958	1011	0.9914	2	5
6	0.1323	17	0.1334	18	7.4947	985	0.9912	2	4
7	0.1340	17	0.1352	18	7.3962	960	0.9910	3	3
8	0.1357	17	0.1370	18	7.3002	936	0.9907	2	2
9	0.1374	18	0.1388	17	7.2066	912	0.9905	2	1
8°.0	0.1392	17	0.1405	18	7.1154	890	0.9903	3	82°.0
1	0.1409	17	0.1423	18	7.0264	869	0.9900	2	9
2	0.1426	18	0.1441	18	6.9395	847	0.9898	3	8
3	0.1444	17	0.1459	18	6.8548	828	0.9895	2	7
4	0.1461	17	0.1477	18	6.7720	808	0.9893	3	6
5	0.1478	17	0.1495	17	6.6912	790	0.9890	2	5
6	0.1495	18	0.1512	18	6.6122	772	0.9888	3	4
7	0.1513	17	0.1530	18	6.5350	754	0.9885	3	3
8	0.1530	17	0.1548	18	6.4596	737	0.9882	2	2
9	0.1547	17	0.1566	18	6.3859	721	0.9880	3	1
9°.0	0.1564		0.1584		6.3138		0.9877		81°.0
	Cos	d	Cot	d	Tan	d	Sin	d	Angle

P P

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

P P

9° to 15°

Angle	Sin	d	Tan	d	Cot	d	Cos	d		P P
9°.0	0.1564	18	0.1584	18	6.3138	706	0.9877	3	81°.0	
1	0.1582	17	0.1602	18	6.2432	690	0.9874	3	9	
2	0.1599	17	0.1620	18	6.1742	676	0.9871	2	8	
3	0.1616	17	0.1638	17	6.1066	661	0.9869	3	7	
4	0.1633	17	0.1655	18	6.0405	647	0.9866	3	6	
5	0.1650	18	0.1673	18	5.9758	634	0.9863	3	5	
6	0.1668	17	0.1691	18	5.9124	622	0.9860	3	4	
7	0.1685	17	0.1709	18	5.8502	608	0.9857	3	3	
8	0.1702	17	0.1727	18	5.7894	597	0.9854	3	2	
9	0.1719	17	0.1745	18	5.7297	584	0.9851	3	1	
10°.0	0.1736	18	0.1763	18	5.6713	573	0.9848	3	80°.0	
1	0.1754	17	0.1781	18	5.6140	562	0.9845	3	9	
2	0.1771	17	0.1799	18	5.5578	552	0.9842	3	8	
3	0.1788	17	0.1817	18	5.5026	540	0.9839	3	7	
4	0.1805	17	0.1835	18	5.4486	531	0.9836	3	6	
5	0.1822	18	0.1853	18	5.3955	520	0.9833	4	5	
6	0.1840	17	0.1871	19	5.3435	511	0.9829	3	4	
7	0.1857	17	0.1890	18	5.2924	502	0.9826	3	3	
8	0.1874	17	0.1908	18	5.2422	493	0.9823	3	2	
9	0.1891	17	0.1926	18	5.1929	483	0.9820	4	1	
11°.0	0.1908	17	0.1944	18	5.1446	476	0.9816	3	79°.0	
1	0.1925	17	0.1962	18	5.0970	466	0.9813	3	9	
2	0.1942	17	0.1980	18	5.0504	459	0.9810	4	8	
3	0.1959	18	0.1998	18	5.0045	451	0.9806	3	7	
4	0.1977	17	0.2016	19	4.9594	442	0.9803	4	6	
5	0.1994	17	0.2035	18	4.9152	436	0.9799	3	5	
6	0.2011	17	0.2053	18	4.8716	428	0.9796	4	4	
7	0.2028	17	0.2071	18	4.8288	421	0.9792	3	3	
8	0.2045	17	0.2089	18	4.7867	414	0.9789	4	2	
9	0.2062	17	0.2107	19	4.7453	407	0.9785	4	1	
12°.0	0.2079	17	0.2126	18	4.7046	400	0.9781	3	78°.0	
1	0.2096	17	0.2144	18	4.6646	394	0.9778	4	9	
2	0.2113	17	0.2162	18	4.6252	388	0.9774	4	8	
3	0.2130	17	0.2180	19	4.5864	381	0.9770	3	7	
4	0.2147	17	0.2199	18	4.5483	376	0.9767	4	6	
5	0.2164	17	0.2217	18	4.5107	370	0.9763	4	5	
6	0.2181	17	0.2235	19	4.4737	364	0.9759	4	4	
7	0.2198	17	0.2254	18	4.4373	358	0.9755	4	3	
8	0.2215	18	0.2272	18	4.4015	353	0.9751	3	2	
9	0.2233	17	0.2290	19	4.3662	347	0.9748	4	1	
13°.0	0.2250	17	0.2309	18	4.3315	343	0.9744	4	77°.0	
1	0.2267	17	0.2327	18	4.2972	337	0.9740	4	9	
2	0.2284	16	0.2345	19	4.2635	332	0.9736	4	8	
3	0.2300	17	0.2364	18	4.2303	327	0.9732	4	7	
4	0.2317	17	0.2382	19	4.1976	323	0.9728	4	6	
5	0.2334	17	0.2401	18	4.1653	318	0.9724	4	5	
6	0.2351	17	0.2419	19	4.1335	313	0.9720	5	4	
7	0.2368	17	0.2438	18	4.1022	309	0.9715	4	3	
8	0.2385	17	0.2456	19	4.0713	305	0.9711	4	2	
9	0.2402	17	0.2475	18	4.0408	300	0.9707	4	1	
14°.0	0.2419	17	0.2493	19	4.0108	296	0.9703	4	76°.0	
1	0.2436	17	0.2512	18	3.9812	292	0.9699	5	9	
2	0.2453	17	0.2530	19	3.9520	288	0.9694	4	8	
3	0.2470	17	0.2549	19	3.9232	285	0.9690	4	7	
4	0.2487	17	0.2568	18	3.8947	280	0.9686	5	6	
5	0.2504	17	0.2586	19	3.8667	276	0.9681	4	5	
6	0.2521	17	0.2605	18	3.8391	273	0.9677	4	4	
7	0.2538	16	0.2623	19	3.8118	270	0.9673	5	3	
8	0.2554	17	0.2642	19	3.7848	265	0.9668	4	2	
9	0.2571	17	0.2661	18	3.7583	262	0.9664	5	1	
15°.0	0.2588		0.2679		3.7321		0.9659		75°.0	
	Cos	d	Cot	d	Tan	d	Sin	d	Angle	P P

P P:

19	
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

18	
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

17	
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

75° to 81°

Angle	Sin	d	Tan	d	Cot	d	Cos	d	
15°.0	0.2588	17	0.2679	19	3.7321	259	0.9659	4	75°.0
1	0.2605	17	0.2698	19	3.7062	256	0.9655	5	9
2	0.2622	17	0.2717	19	3.6806	252	0.9650	4	8
3	0.2639	17	0.2736	18	3.6554	249	0.9646	5	7
4	0.2656	16	0.2754	19	3.6305	246	0.9641	5	6
5	0.2672	17	0.2773	19	3.6059	243	0.9636	4	5
6	0.2689	17	0.2792	19	3.5816	240	0.9632	5	4
7	0.2706	17	0.2811	19	3.5576	237	0.9627	5	3
8	0.2723	17	0.2830	19	3.5339	234	0.9622	5	2
9	0.2740	16	0.2849	18	3.5105	231	0.9617	4	1
16°.0	0.2756	17	0.2867	19	3.4874	228	0.9613	5	74°.0
1	0.2773	17	0.2886	19	3.4646	226	0.9608	5	9
2	0.2790	17	0.2905	19	3.4420	223	0.9603	5	8
3	0.2807	16	0.2924	19	3.4197	220	0.9598	5	7
4	0.2823	17	0.2943	19	3.3977	218	0.9593	5	6
5	0.2840	17	0.2962	19	3.3759	215	0.9588	5	5
6	0.2857	17	0.2981	19	3.3544	212	0.9583	5	4
7	0.2874	16	0.3000	19	3.3332	210	0.9578	5	3
8	0.2890	17	0.3019	19	3.3122	208	0.9573	5	2
9	0.2907	17	0.3038	19	3.2914	205	0.9568	5	1
17°.0	0.2924	16	0.3057	19	3.2709	203	0.9563	5	73°.0
1	0.2940	17	0.3076	20	3.2506	201	0.9558	5	9
2	0.2957	17	0.3096	19	3.2305	199	0.9553	5	8
3	0.2974	16	0.3115	19	3.2106	196	0.9548	6	7
4	0.2990	17	0.3134	19	3.1910	194	0.9542	5	6
5	0.3007	17	0.3153	19	3.1716	192	0.9537	5	5
6	0.3024	16	0.3172	19	3.1524	190	0.9532	5	4
7	0.3040	17	0.3191	20	3.1334	188	0.9527	6	3
8	0.3057	17	0.3211	19	3.1146	185	0.9521	5	2
9	0.3074	16	0.3230	19	3.0961	184	0.9516	5	1
18°.0	0.3090	17	0.3249	20	3.0777	182	0.9511	6	72°.0
1	0.3107	16	0.3269	19	3.0595	180	0.9505	5	9
2	0.3123	17	0.3288	19	3.0415	178	0.9500	6	8
3	0.3140	16	0.3307	20	3.0237	176	0.9494	5	7
4	0.3156	17	0.3327	19	3.0061	174	0.9489	6	6
5	0.3173	17	0.3346	19	2.9887	173	0.9483	5	5
6	0.3190	16	0.3365	20	2.9714	170	0.9478	6	4
7	0.3206	17	0.3385	19	2.9544	169	0.9472	6	3
8	0.3223	16	0.3404	20	2.9375	167	0.9466	5	2
9	0.3239	17	0.3424	19	2.9208	166	0.9461	6	1
19°.0	0.3256	16	0.3443	20	2.9042	164	0.9455	6	71°.0
1	0.3272	17	0.3463	19	2.8878	162	0.9449	5	9
2	0.3289	16	0.3482	20	2.8716	160	0.9444	6	8
3	0.3305	17	0.3502	20	2.8556	159	0.9438	6	7
4	0.3322	16	0.3522	19	2.8397	158	0.9432	6	6
5	0.3338	17	0.3541	20	2.8239	156	0.9426	5	5
6	0.3355	16	0.3561	20	2.8083	154	0.9421	6	4
7	0.3371	16	0.3581	19	2.7929	153	0.9415	6	3
8	0.3387	17	0.3600	20	2.7776	151	0.9409	6	2
9	0.3404	16	0.3620	20	2.7625	150	0.9403	6	1
20°.0	0.3420	17	0.3640	19	2.7475	149	0.9397	6	70°.0
1	0.3437	16	0.3659	20	2.7326	147	0.9391	6	9
2	0.3453	16	0.3679	20	2.7179	145	0.9385	6	8
3	0.3469	17	0.3699	20	2.7034	145	0.9379	6	7
4	0.3486	16	0.3719	20	2.6889	143	0.9373	6	6
5	0.3502	16	0.3739	20	2.6746	141	0.9367	6	5
6	0.3518	17	0.3759	20	2.6605	141	0.9361	7	4
7	0.3535	16	0.3779	20	2.6464	139	0.9354	6	3
8	0.3551	16	0.3799	20	2.6325	138	0.9348	6	2
9	0.3567	17	0.3819	20	2.6187	136	0.9342	6	1
21°.0	0.3584		0.3839		2.6051		0.9336		69°.0
	Cos	d	Cot	d	Tan	d	Sin	d	Angle

P P

	20
1	2.0
2	4.0
3	6.0
4	8.0
5	10.0
6	12.0
7	14.0
8	16.0
9	18.0

	19
1	1.9
2	3.8
3	5.7
4	7.6
5	9.5
6	11.4
7	13.3
8	15.2
9	17.1

	18
1	1.8
2	3.6
3	5.4
4	7.2
5	9.0
6	10.8
7	12.6
8	14.4
9	16.2

	17
1	1.7
2	3.4
3	5.1
4	6.8
5	8.5
6	10.2
7	11.9
8	13.6
9	15.3

	16
1	1.6
2	3.2
3	4.8
4	6.4
5	8.0
6	9.6
7	11.2
8	12.8
9	14.4

P P

Angle	Sin	d	Tan	d	Cot	d	Cos	d	Angle
21°.0	0.3584		0.3839		2.6051		0.9336		69°.0
1	0.3600	16	0.3859	20	2.5916	135	0.9330	6	9
2	0.3616	16	0.3879	20	2.5782	134	0.9323	7	8
3	0.3633	17	0.3899	20	2.5649	133	0.9317	6	7
4	0.3649	16	0.3919	20	2.5517	132	0.9311	6	6
5	0.3665	16	0.3939	20	2.5386	131	0.9304	7	5
6	0.3681	16	0.3959	20	2.5257	129	0.9298	6	4
7	0.3697	16	0.3979	20	2.5129	128	0.9291	7	3
8	0.3714	17	0.4000	21	2.5002	127	0.9285	6	2
9	0.3730	16	0.4020	20	2.4876	126	0.9278	7	1
22°.0	0.3746	16	0.4040	20	2.4751	125	0.9272	6	68°.0
1	0.3762	16	0.4061	21	2.4627	124	0.9265	7	9
2	0.3778	16	0.4081	20	2.4504	123	0.9259	6	8
3	0.3795	17	0.4101	20	2.4383	121	0.9252	7	7
4	0.3811	16	0.4122	21	2.4262	121	0.9245	7	6
5	0.3827	16	0.4142	20	2.4142	120	0.9239	6	5
6	0.3843	16	0.4163	21	2.4023	119	0.9232	7	4
7	0.3859	16	0.4183	20	2.3906	117	0.9225	7	3
8	0.3875	16	0.4204	21	2.3789	117	0.9219	6	2
9	0.3891	16	0.4224	20	2.3673	116	0.9212	7	1
23°.0	0.3907	16	0.4245	21	2.3559	114	0.9205	7	67°.0
1	0.3923	16	0.4265	20	2.3445	114	0.9198	7	9
2	0.3939	16	0.4286	21	2.3332	113	0.9191	7	8
3	0.3955	16	0.4307	21	2.3220	112	0.9184	7	7
4	0.3971	16	0.4327	20	2.3109	111	0.9178	6	6
5	0.3987	16	0.4348	21	2.2998	111	0.9171	7	5
6	0.4003	16	0.4369	21	2.2889	109	0.9164	7	4
7	0.4019	16	0.4390	21	2.2781	108	0.9157	7	3
8	0.4035	16	0.4411	21	2.2673	108	0.9150	7	2
9	0.4051	16	0.4431	20	2.2566	107	0.9143	7	1
24°.0	0.4067	16	0.4452	21	2.2460	106	0.9135	8	66°.0
1	0.4083	16	0.4473	21	2.2355	105	0.9128	7	9
2	0.4099	16	0.4494	21	2.2251	104	0.9121	7	8
3	0.4115	16	0.4515	21	2.2148	103	0.9114	7	7
4	0.4131	16	0.4536	21	2.2045	103	0.9107	7	6
5	0.4147	16	0.4557	21	2.1943	102	0.9100	7	5
6	0.4163	16	0.4578	21	2.1842	101	0.9092	8	4
7	0.4179	16	0.4599	21	2.1742	100	0.9085	7	3
8	0.4195	16	0.4621	22	2.1642	100	0.9078	7	2
9	0.4210	15	0.4642	21	2.1543	99	0.9070	8	1
25°.0	0.4226	16	0.4663	21	2.1445	98	0.9063	7	65°.0
1	0.4242	16	0.4684	21	2.1348	97	0.9056	7	9
2	0.4258	16	0.4706	22	2.1251	97	0.9048	8	8
3	0.4274	16	0.4727	21	2.1155	96	0.9041	7	7
4	0.4289	15	0.4748	21	2.1060	95	0.9033	8	6
5	0.4305	16	0.4770	22	2.0965	95	0.9026	7	5
6	0.4321	16	0.4791	21	2.0872	93	0.9018	8	4
7	0.4337	16	0.4813	22	2.0778	94	0.9011	7	3
8	0.4352	15	0.4834	21	2.0686	92	0.9003	8	2
9	0.4368	16	0.4856	22	2.0594	92	0.8996	7	1
26°.0	0.4384	16	0.4877	21	2.0503	91	0.8988	8	64°.0
1	0.4399	15	0.4899	22	2.0413	90	0.8980	8	9
2	0.4415	16	0.4921	22	2.0323	90	0.8973	7	8
3	0.4431	16	0.4942	21	2.0233	90	0.8965	8	7
4	0.4446	15	0.4964	22	2.0145	88	0.8957	8	6
5	0.4462	16	0.4986	22	2.0057	88	0.8949	8	5
6	0.4478	16	0.5008	22	1.9970	87	0.8942	7	4
7	0.4493	15	0.5029	21	1.9883	87	0.8934	8	3
8	0.4509	16	0.5051	22	1.9797	86	0.8926	8	2
9	0.4524	15	0.5073	22	1.9711	86	0.8918	8	1
27°.0	0.4540	16	0.5095	22	1.9626	85	0.8910	8	63°.0
	Cos	d	Cot	d	Tan	d	Sin	d	Angle

P P

	22	21	20	17	16	15
1	2.2	2.1	2.0	1.7	1.6	1.5
2	4.4	4.2	4.0	3.4	3.2	3.0
3	6.6	6.3	6.0	5.1	4.8	4.5
4	8.8	8.4	8.0	6.8	6.4	6.0
5	11.0	10.5	10.0	8.5	8.0	7.5
6	13.2	12.6	12.0	10.2	9.6	9.0
7	15.4	14.7	14.0	11.9	11.2	10.5
8	17.6	16.8	16.0	13.6	12.8	12.0
9	19.8	18.9	18.0	15.3	14.4	13.5

Angle	Sin	d	Tan	d	Cot	d	Cos	d		P P
27°.0	0.4540	15	0.5095	22	1.9626	84	0.8910	8	63°.0	
1	0.4555	16	0.5117	22	1.9542	84	0.8902	8	9	25
2	0.4571	15	0.5139	22	1.9458	83	0.8894	8	8	1 | 2.5
3	0.4586	16	0.5161	23	1.9375	83	0.8886	8	7	2 | 5.0
4	0.4602	15	0.5184	22	1.9292	82	0.8878	8	6	3 | 7.5
5	0.4617	16	0.5206	22	1.9210	82	0.8870	8	5	4 | 10.0
6	0.4633	15	0.5228	22	1.9128	81	0.8862	8	4	5 | 12.5
7	0.4648	16	0.5250	22	1.9047	80	0.8854	8	3	6 | 15.0
8	0.4664	15	0.5272	23	1.8967	80	0.8846	8	2	7 | 17.5
9	0.4679	16	0.5295	22	1.8887	80	0.8838	9	1	8 | 20.0
28°.0	0.4695	15	0.5317	23	1.8807	79	0.8829	8	62°.0	9 | 22.5
1	0.4710	16	0.5340	22	1.8728	78	0.8821	8	9	24
2	0.4726	15	0.5362	22	1.8650	78	0.8813	8	8	1 | 2.4
3	0.4741	15	0.5384	23	1.8572	77	0.8805	9	7	2 | 4.8
4	0.4756	16	0.5407	23	1.8495	77	0.8796	8	6	3 | 7.2
5	0.4772	15	0.5430	22	1.8418	77	0.8788	8	5	4 | 9.6
6	0.4787	15	0.5452	23	1.8341	76	0.8780	9	4	5 | 12.0
7	0.4802	16	0.5475	23	1.8265	75	0.8771	8	3	6 | 14.4
8	0.4818	15	0.5498	22	1.8190	75	0.8763	8	2	7 | 16.8
9	0.4833	15	0.5520	23	1.8115	75	0.8755	9	1	8 | 19.2
29°.0	0.4848	15	0.5543	23	1.8040	74	0.8746	8	61°.0	9 | 21.6
1	0.4863	16	0.5566	23	1.7966	73	0.8738	9	9	23
2	0.4879	15	0.5589	23	1.7893	73	0.8729	8	8	1 | 2.3
3	0.4894	15	0.5612	23	1.7820	73	0.8721	9	7	2 | 4.6
4	0.4909	15	0.5635	23	1.7747	72	0.8712	8	6	3 | 6.9
5	0.4924	15	0.5658	23	1.7675	72	0.8704	9	5	4 | 9.2
6	0.4939	16	0.5681	23	1.7603	71	0.8695	9	4	5 | 11.5
7	0.4955	15	0.5704	23	1.7532	71	0.8686	8	3	6 | 13.8
8	0.4970	15	0.5727	23	1.7461	70	0.8678	9	2	7 | 16.1
9	0.4985	15	0.5750	24	1.7391	70	0.8669	9	1	8 | 18.4
30°.0	0.5000	15	0.5774	23	1.7321	70	0.8660	8	60°.0	9 | 20.7
1	0.5015	15	0.5797	23	1.7251	69	0.8652	9	9	22
2	0.5030	15	0.5820	24	1.7182	69	0.8643	9	8	1 | 2.2
3	0.5045	15	0.5844	23	1.7113	68	0.8634	9	7	2 | 4.4
4	0.5060	15	0.5867	23	1.7045	68	0.8625	9	6	3 | 6.6
5	0.5075	15	0.5890	24	1.6977	68	0.8616	9	5	4 | 8.8
6	0.5090	15	0.5914	24	1.6909	67	0.8607	8	4	5 | 11.0
7	0.5105	15	0.5938	23	1.6842	67	0.8599	9	3	6 | 13.2
8	0.5120	15	0.5961	24	1.6775	66	0.8590	9	2	7 | 15.4
9	0.5135	15	0.5985	24	1.6709	66	0.8581	9	1	8 | 17.6
31°.0	0.5150	15	0.6009	23	1.6643	66	0.8572	9	59°.0	9 | 19.8
1	0.5165	15	0.6032	24	1.6577	65	0.8563	9	9	16
2	0.5180	15	0.6056	24	1.6512	65	0.8554	9	8	1 | 1.6
3	0.5195	15	0.6080	24	1.6447	64	0.8545	9	7	2 | 3.2
4	0.5210	15	0.6104	24	1.6383	64	0.8536	10	6	3 | 4.8
5	0.5225	15	0.6128	24	1.6319	64	0.8526	9	5	4 | 6.4
6	0.5240	15	0.6152	24	1.6255	64	0.8517	9	4	5 | 8.0
7	0.5255	15	0.6176	24	1.6191	63	0.8508	9	3	6 | 9.6
8	0.5270	14	0.6200	24	1.6128	62	0.8499	9	2	7 | 11.2
9	0.5284	15	0.6224	25	1.6066	63	0.8490	10	1	8 | 12.8
32°.0	0.5299	15	0.6249	24	1.6003	62	0.8480	9	58°.0	9 | 14.4
1	0.5314	15	0.6273	24	1.5941	61	0.8471	9	9	15
2	0.5329	15	0.6297	25	1.5880	62	0.8462	9	8	1 | 1.5
3	0.5344	14	0.6322	24	1.5818	61	0.8453	10	7	2 | 3.0
4	0.5358	15	0.6346	25	1.5757	60	0.8443	9	6	3 | 4.5
5	0.5373	15	0.6371	24	1.5697	60	0.8434	9	5	4 | 6.0
6	0.5388	14	0.6395	25	1.5637	60	0.8425	10	4	5 | 7.5
7	0.5402	15	0.6420	25	1.5577	60	0.8415	9	3	6 | 9.0
8	0.5417	15	0.6445	24	1.5517	59	0.8406	10	2	7 | 10.5
9	0.5432	14	0.6469	25	1.5458	59	0.8396	9	1	8 | 12.0
33.°0	0.5446		0.6494		1.5399		0.8387		57°.0	9 | 13.5
	Cos	d	Cot	d	Tan	d	Sin	d	Angle	P P

Angle	Sin	d	Tan	d	Cot	d	Cos	d		P P
33°.0	0.5446	15	0.6494	25	1.5399	59	0.8387	10	57°.0	
1	0.5461	15	0.6519	25	1.5340	58	0.8377	9	9	
2	0.5476	14	0.6544	25	1.5282	58	0.8368	10	8	
3	0.5490	15	0.6569	25	1.5224	58	0.8358	10	7	
4	0.5505	14	0.6594	25	1.5166	58	0.8348	9	6	
5	0.5519	15	0.6619	25	1.5108	57	0.8339	10	5	
6	0.5534	14	0.6644	25	1.5051	57	0.8329	9	4	
7	0.5548	15	0.6669	25	1.4994	56	0.8320	10	3	
8	0.5563	14	0.6694	26	1.4938	56	0.8310	10	2	
9	0.5577	15	0.6720	25	1.4882	56	0.8300	10	1	
34°.0	0.5592	14	0.6745	26	1.4826	56	0.8290	9	56°.0	
1	0.5606	15	0.6771	25	1.4770	55	0.8281	10	9	
2	0.5621	14	0.6796	26	1.4715	56	0.8271	10	8	
3	0.5635	15	0.6822	25	1.4659	54	0.8261	10	7	
4	0.5650	14	0.6847	26	1.4605	55	0.8251	10	6	
5	0.5664	14	0.6873	26	1.4550	54	0.8241	10	5	
6	0.5678	15	0.6899	25	1.4496	54	0.8231	10	4	
7	0.5693	14	0.6924	26	1.4442	54	0.8221	10	3	
8	0.5707	14	0.6950	26	1.4388	53	0.8211	9	2	
9	0.5721	15	0.6976	26	1.4335	54	0.8202	10	1	
35°.0	0.5736	14	0.7002	26	1.4281	52	0.8192	11	55°.0	
1	0.5750	14	0.7028	26	1.4229	53	0.8181	10	9	
2	0.5764	15	0.7054	26	1.4176	52	0.8171	10	8	
3	0.5779	14	0.7080	27	1.4124	53	0.8161	10	7	
4	0.5793	14	0.7107	26	1.4071	52	0.8151	10	6	
5	0.5807	14	0.7133	26	1.4019	51	0.8141	10	5	
6	0.5821	14	0.7159	27	1.3968	52	0.8131	10	4	
7	0.5835	15	0.7186	26	1.3916	51	0.8121	10	3	
8	0.5850	14	0.7212	27	1.3865	51	0.8111	11	2	
9	0.5864	14	0.7239	26	1.3814	50	0.8100	10	1	
36°.0	0.5878	14	0.7265	27	1.3764	51	0.8090	10	54°.0	
1	0.5892	14	0.7292	27	1.3713	50	0.8080	10	9	
2	0.5906	14	0.7319	27	1.3663	50	0.8070	11	8	
3	0.5920	14	0.7346	27	1.3613	49	0.8059	10	7	
4	0.5934	14	0.7373	27	1.3564	50	0.8049	10	6	
5	0.5948	14	0.7400	27	1.3514	49	0.8039	11	5	
6	0.5962	14	0.7427	27	1.3465	49	0.8028	10	4	
7	0.5976	14	0.7454	27	1.3416	49	0.8018	11	3	
8	0.5990	14	0.7481	27	1.3367	48	0.8007	10	2	
9	0.6004	14	0.7508	28	1.3319	49	0.7997	11	1	
37°.0	0.6018	14	0.7536	27	1.3270	48	0.7986	10	53°.0	
1	0.6032	14	0.7563	27	1.3222	47	0.7976	11	9	
2	0.6046	14	0.7590	28	1.3175	48	0.7965	10	8	
3	0.6060	14	0.7618	28	1.3127	48	0.7955	11	7	
4	0.6074	14	0.7646	27	1.3079	47	0.7944	10	6	
5	0.6088	13	0.7673	28	1.3032	47	0.7934	11	5	
6	0.6101	14	0.7701	28	1.2985	47	0.7923	11	4	
7	0.6115	14	0.7729	28	1.2938	46	0.7912	10	3	
8	0.6129	14	0.7757	28	1.2892	46	0.7902	11	2	
9	0.6143	14	0.7785	28	1.2846	47	0.7891	11	1	
38°.0	0.6157	13	0.7813	28	1.2799	46	0.7880	11	52°.0	
1	0.6170	14	0.7841	28	1.2753	45	0.7869	10	9	
2	0.6184	14	0.7869	29	1.2708	46	0.7859	11	8	
3	0.6198	13	0.7898	28	1.2662	45	0.7848	11	7	
4	0.6211	14	0.7926	28	1.2617	45	0.7837	11	6	
5	0.6225	14	0.7954	29	1.2572	45	0.7826	11	5	
6	0.6239	13	0.7983	29	1.2527	45	0.7815	11	4	
7	0.6252	14	0.8012	28	1.2482	45	0.7804	11	3	
8	0.6266	14	0.8040	29	1.2437	44	0.7793	11	2	
9	0.6280	13	0.8069	29	1.2393	44	0.7782	11	1	
39°.0	0.6293		0.8098		1.2349		0.7771		51°.0	
	Cos	d	Cot	d	Tan	d	Sin	d	Angle	P P

P P

	29	28	27	26	15	14	13
1	2.9	2.8	2.7	2.6	1.5	1.4	1.3
2	5.8	5.6	5.4	5.2	3.0	2.8	2.6
3	8.7	8.4	8.1	7.8	4.5	4.2	3.9
4	11.6	11.2	10.8	10.4	6.0	5.6	5.2
5	14.5	14.0	13.5	13.0	7.5	7.0	6.5
6	17.4	16.8	16.2	15.6	9.0	8.4	7.8
7	20.3	19.6	18.9	18.2	10.5	9.8	9.1
8	23.2	22.4	21.6	20.8	12.0	11.2	10.4
9	26.1	25.2	24.3	23.4	13.5	12.6	11.7

Angle	Sin	d	Tan	d	Cot	d	Cos	d	
39°.0	0.6293	14	0.8098	29	1.2349	44	0.7771	11	51°.0
1	0.6307	13	0.8127	29	1.2305	44	0.7760	11	9
2	0.6320	14	0.8156	29	1.2261	43	0.7749	11	8
3	0.6334	13	0.8185	29	1.2218	44	0.7738	11	7
4	0.6347	14	0.8214	29	1.2174	43	0.7727	11	6
5	0.6361	13	0.8243	30	1.2131	43	0.7716	11	5
6	0.6374	14	0.8273	29	1.2088	43	0.7705	11	4
7	0.6388	13	0.8302	30	1.2045	43	0.7694	11	3
8	0.6401	13	0.8332	29	1.2002	42	0.7683	11	2
9	0.6414	14	0.8361	30	1.1960	42	0.7672	12	1
40°.0	0.6428	13	0.8391	30	1.1918	43	0.7660	11	50°.0
1	0.6441	14	0.8421	30	1.1875	42	0.7649	11	9
2	0.6455	13	0.8451	30	1.1833	41	0.7638	11	8
3	0.6468	13	0.8481	30	1.1792	42	0.7627	12	7
4	0.6481	13	0.8511	30	1.1750	42	0.7615	11	6
5	0.6494	14	0.8541	30	1.1708	41	0.7604	11	5
6	0.6508	13	0.8571	30	1.1667	41	0.7593	12	4
7	0.6521	13	0.8601	31	1.1626	41	0.7581	11	3
8	0.6534	13	0.8632	30	1.1585	41	0.7570	11	2
9	0.6547	14	0.8662	31	1.1544	40	0.7559	12	1
41°.0	0.6561	13	0.8693	31	1.1504	41	0.7547	11	49°.0
1	0.6574	13	0.8724	30	1.1463	40	0.7536	12	9
2	0.6587	13	0.8754	31	1.1423	40	0.7524	11	8
3	0.6600	13	0.8785	31	1.1383	40	0.7513	12	7
4	0.6613	13	0.8816	31	1.1343	40	0.7501	11	6
5	0.6626	13	0.8847	31	1.1303	40	0.7490	12	5
6	0.6639	13	0.8878	32	1.1263	39	0.7478	12	4
7	0.6652	13	0.8910	31	1.1224	40	0.7466	11	3
8	0.6665	13	0.8941	31	1.1184	39	0.7455	12	2
9	0.6678	13	0.8972	32	1.1145	39	0.7443	12	1
42°.0	0.6691	13	0.9004	32	1.1106	39	0.7431	11	48°.0
1	0.6704	13	0.9036	31	1.1067	39	0.7420	12	9
2	0.6717	13	0.9067	32	1.1028	38	0.7408	12	8
3	0.6730	13	0.9099	32	1.0990	39	0.7396	11	7
4	0.6743	13	0.9131	32	1.0951	38	0.7385	12	6
5	0.6756	13	0.9163	32	1.0913	38	0.7373	12	5
6	0.6769	13	0.9195	33	1.0875	38	0.7361	12	4
7	0.6782	12	0.9228	32	1.0837	38	0.7349	12	3
8	0.6794	13	0.9260	33	1.0799	38	0.7337	12	2
9	0.6807	13	0.9293	32	1.0761	37	0.7325	11	1
43°.0	0.6820	13	0.9325	33	1.0724	38	0.7314	12	47°.0
1	0.6833	12	0.9358	33	1.0686	37	0.7302	12	9
2	0.6845	13	0.9391	33	1.0649	37	0.7290	12	8
3	0.6858	13	0.9424	33	1.0612	37	0.7278	12	7
4	0.6871	13	0.9457	33	1.0575	37	0.7266	12	6
5	0.6884	12	0.9490	33	1.0538	37	0.7254	12	5
6	0.6896	13	0.9523	33	1.0501	37	0.7242	12	4
7	0.6909	12	0.9556	34	1.0464	36	0.7230	12	3
8	0.6921	13	0.9590	33	1.0428	36	0.7218	12	2
9	0.6934	13	0.9623	34	1.0392	37	0.7206	13	1
44°.0	0.6947	12	0.9657	34	1.0355	36	0.7193	12	46°.0
1	0.6959	13	0.9691	34	1.0319	36	0.7181	12	9
2	0.6972	12	0.9725	34	1.0283	36	0.7169	12	8
3	0.6984	13	0.9759	34	1.0247	35	0.7157	12	7
4	0.6997	12	0.9793	34	1.0212	36	0.7145	12	6
5	0.7009	13	0.9827	34	1.0176	35	0.7133	13	5
6	0.7022	12	0.9861	35	1.0141	36	0.7120	12	4
7	0.7034	12	0.9896	34	1.0105	35	0.7108	12	3
8	0.7046	13	0.9930	35	1.0070	35	0.7096	13	2
9	0.7059	12	0.9965	35	1.0035	35	0.7083	12	1
45°.0	0.7071		1.0000		1.0000		0.7071		45°.0
	Cos	d	Cot	d	Tan	d	Sin	d	Angle

P P

	34	33	32	31	29	14	13
1	3.4	3.3	3.2	3.1	2.9	1.4	1.3
2	6.8	6.6	6.4	6.2	5.8	2.8	2.6
3	10.2	9.9	9.6	9.3	8.7	4.2	3.9
4	13.6	13.2	12.8	12.4	11.6	5.6	5.2
5	17.0	16.5	16.0	15.5	14.5	7.0	6.5
6	20.4	19.8	19.2	18.6	17.4	8.4	7.8
7	23.8	23.1	22.4	21.7	20.3	9.8	9.1
8	27.2	26.4	25.6	24.8	23.2	11.2	10.4
9	30.6	29.7	28.8	27.9	26.1	12.6	11.7

TABLE VI

FOUR-PLACE VALUES

OF THE

SQUARES OF NUMBERS

FROM 0.000 TO 3.500

Squares of Numbers from 0.000 to 0.500

N	N^2 0	1	2	3	4	5	6	7	8	9
0.00	0.0000	0000	0000	0000	0000	0000	0000	0000	0001	0001
01	0001	0001	0001	0002	0002	0002	0003	0003	0003	0004
02	0004	0004	0005	0005	0006	0006	0007	0007	0008	0008
03	0009	0010	0010	0011	0012	0012	0013	0014	0014	0015
04	0016	0017	0018	0018	0019	0020	0021	0022	0023	0024
05	0.0025	0026	0027	0028	0029	0030	0031	0032	0034	0035
06	0036	0037	0038	0040	0041	0042	0044	0045	0046	0048
07	0049	0050	0052	0053	0055	0056	0058	0059	0061	0062
08	0064	0066	0067	0069	0071	0072	0074	0076	0077	0079
09	0081	0083	0085	0086	0088	0090	0092	0094	0096	0098
0.10	0.0100	0102	0104	0106	0108	0110	0112	0114	0117	0119
11	0121	0123	0125	0128	0130	0132	0135	0137	0139	0142
12	0144	0146	0149	0151	0154	0156	0159	0161	0164	0166
13	0169	0172	0174	0177	0180	0182	0185	0188	0190	0193
14	0196	0199	0202	0204	0207	0210	0213	0216	0219	0222
15	0.0225	0228	0231	0234	0237	0240	0243	0246	0250	0253
16	0256	0259	0262	0266	0269	0272	0276	0279	0282	0286
17	0289	0292	0296	0299	0303	0306	0310	0313	0317	0320
18	0324	0328	0331	0335	0339	0342	0346	0350	0353	0357
19	0361	0365	0369	0372	0376	0380	0384	0388	0392	0396
0.20	0.0400	0404	0408	0412	0416	0420	0424	0428	0433	0437
21	0441	0445	0449	0454	0458	0462	0467	0471	0475	0480
22	0484	0488	0493	0497	0502	0506	0511	0515	0520	0524
23	0529	0534	0538	0543	0548	0552	0557	0562	0566	0571
24	0576	0581	0586	0590	0595	0600	0605	0610	0615	0620
25	0.0625	0630	0635	0640	0645	0650	0655	0660	0666	0671
26	0676	0681	0686	0692	0697	0702	0708	0713	0718	0724
27	0729	0734	0740	0745	0751	0756	0762	0767	0773	0778
28	0784	0790	0795	0801	0807	0812	0818	0824	0829	0835
29	0841	0847	0853	0858	0864	0870	0876	0882	0888	0894
0.30	0.0900	0906	0912	0918	0924	0930	0936	0942	0949	0955
31	0961	0967	0973	0980	0986	0992	0999	1005	1011	1018
32	1024	1030	1037	1043	1050	1056	1063	1069	1076	1082
33	1089	1096	1102	1109	1116	1122	1129	1136	1142	1149
34	1156	1163	1170	1176	1183	1190	1197	1204	1211	1218
35	0.1225	1232	1239	1246	1253	1260	1267	1274	1282	1289
36	1296	1303	1310	1318	1325	1332	1340	1347	1354	1362
37	1369	1376	1384	1391	1399	1406	1414	1421	1429	1436
38	1444	1452	1459	1467	1475	1482	1490	1498	1505	1513
39	1521	1529	1537	1544	1552	1560	1568	1576	1584	1592
0.40	0.1600	1608	1616	1624	1632	1640	1648	1656	1665	1673
41	1681	1689	1697	1706	1714	1722	1731	1739	1747	1756
42	1764	1772	1781	1789	1798	1806	1815	1823	1832	1840
43	1849	1858	1866	1875	1884	1892	1901	1910	1918	1927
44	1936	1945	1954	1962	1971	1980	1989	1998	2007	2016
45	0.2025	2034	2043	2052	2061	2070	2079	2088	2098	2107
46	2116	2125	2134	2144	2153	2162	2172	2181	2190	2200
47	2209	2218	2228	2237	2247	2256	2266	2275	2285	2294
48	2304	2314	2323	2333	2343	2352	2362	2372	2381	2391
49	2401	2411	2421	2430	2440	2450	2460	2470	2480	2490
0.50	0.2500	2510	2520	2530	2540	2550	2560	2570	2581	2591
N	N^2 0	1	2	3	4	5	6	7	8	9

P P

	2	3	4	5	6	7	8	9	10	11
1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1
2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7	3.0	3.3
4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4
5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5
6	1.2	1.8	2.4	3.0	3.6	4.2	4.8	5.4	6.0	6.6
7	1.4	2.1	2.8	3.5	4.2	4.9	5.6	6.3	7.0	7.7
8	1.6	2.4	3.2	4.0	4.8	5.6	6.4	7.2	8.0	8.8
9	1.8	2.7	3.6	4.5	5.4	6.3	7.2	8.1	9.0	9.9

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N^2.

Squares of Numbers from 0.500 to 1.000

N	N² 0	1	2	3	4	5	6	7	8	9
0.50	0.2500	2510	2520	2530	2540	2550	2560	2570	2581	2591
51	2601	2611	2621	2632	2642	2652	2663	2673	2683	2694
52	2704	2714	2725	2735	2746	2756	2767	2777	2788	2798
53	2809	2820	2830	2841	2852	2862	2873	2884	2894	2905
54	2916	2927	2938	2948	2959	2970	2981	2992	3003	3014
55	0.3025	3036	3047	3058	3069	3080	3091	3102	3114	3125
56	3136	3147	3158	3170	3181	3192	3204	3215	3226	3238
57	3249	3260	3272	3283	3295	3306	3318	3329	3341	3352
58	3364	3376	3387	3399	3411	3422	3434	3446	3457	3469
59	3481	3493	3505	3516	3528	3540	3552	3564	3576	3588
0.60	0.3600	3612	3624	3636	3648	3660	3672	3684	3697	3709
61	3721	3733	3745	3758	3770	3782	3795	3807	3819	3832
62	3844	3856	3869	3881	3894	3906	3919	3931	3944	3956
63	3969	3982	3994	4007	4020	4032	4045	4058	4070	4083
64	4096	4109	4122	4134	4147	4160	4173	4186	4199	4212
65	0.4225	4238	4251	4264	4277	4290	4303	4316	4330	4343
66	4356	4369	4382	4396	4409	4422	4436	4449	4462	4476
67	4489	4502	4516	4529	4543	4556	4570	4583	4597	4610
68	4624	4638	4651	4665	4679	4692	4706	4720	4733	4747
69	4761	4775	4789	4802	4816	4830	4844	4858	4872	4886
0.70	0.4900	4914	4928	4942	4956	4970	4984	4998	5013	5027
71	5041	5055	5069	5084	5098	5112	5127	5141	5155	5170
72	5184	5198	5213	5227	5242	5256	5271	5285	5300	5314
73	5329	5344	5358	5373	5388	5402	5417	5432	5446	5461
74	5476	5491	5506	5520	5535	5550	5565	5580	5595	5610
75	0.5625	5640	5655	5670	5685	5700	5715	5730	5746	5761
76	5776	5791	5806	5822	5837	5852	5868	5883	5898	5914
77	5929	5944	5960	5975	5991	6006	6022	6037	6053	6068
78	6084	6100	6115	6131	6147	6162	6178	6194	6209	6225
79	6241	6257	6273	6288	6304	6320	6336	6352	6368	6384
0.80	0.6400	6416	6432	6448	6464	6480	6496	6512	6529	6545
81	6561	6577	6593	6610	6626	6642	6659	6675	6691	6708
82	6724	6740	6757	6773	6790	6806	6823	6839	6856	6872
83	6889	6906	6922	6939	6956	6972	6989	7006	7022	7039
84	7056	7073	7090	7106	7123	7140	7157	7174	7191	7208
85	0.7225	7242	7259	7276	7293	7310	7327	7344	7362	7379
86	7396	7413	7430	7448	7465	7482	7500	7517	7534	7552
87	7569	7586	7604	7621	7639	7656	7674	7691	7709	7726
88	7744	7762	7779	7797	7815	7832	7850	7868	7885	7903
89	7921	7939	7957	7974	7992	8010	8028	8046	8064	8082
0.90	0.8100	8118	8136	8154	8172	8190	8208	8226	8245	8263
91	8281	8299	8317	8336	8354	8372	8391	8409	8427	8446
92	8464	8482	8501	8519	8538	8556	8575	8593	8612	8630
93	8649	8668	8686	8705	8724	8742	8761	8780	8798	8817
94	8836	8855	8874	8892	8911	8930	8949	8968	8987	9006
95	0.9025	9044	9063	9082	9101	9120	9139	9158	9178	9197
96	9216	9235	9254	9274	9293	9312	9332	9351	9370	9390
97	9409	9428	9448	9467	9487	9506	9526	9545	9565	9584
98	9604	9624	9643	9663	9683	9702	9722	9742	9761	9781
99	0.9801	9821	9841	9860	9880	9900	9920	9940	9960	9980
1.00	1.0000	0020	0040	0060	0080	0100	0120	0140	0161	0181
N	N² 0	1	2	3	4	5	6	7	8	9

PP

	10	11	12	13	14	15	16	17	18	19	20	21
1	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1
2	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0	4.2
3	3.0	3.3	3.6	3.9	4.2	4.5	4.8	5.1	5.4	5.7	6.0	6.3
4	4.0	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7,6	8.0	8.4
5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5
6	6.0	6.6	7.2	7.8	8.4	9.0	9.6	10.2	10.8	11.4	12.0	12.6
7	7.0	7.7	8.4	9.1	9.8	10.5	11.2	11.9	12.6	13.3	14.0	14.7
8	8.0	8.8	9.6	10.4	11.2	12.0	12.8	13.6	14.4	15.2	16.0	16.8
9	9.0	9.9	10.8	11.7	12.6	13:5	14.4	15.3	16.2	17.1	18.0	18.9

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N².

Squares of Numbers from 1.000 to 1.500

N	N² 0	1	2	3	4	5	6	7	8	9
1.00	1.0000	0020	0040	0060	0089	0100	0120	0140	0161	0181
01	0201	0221	0241	0262	0282	0302	0323	0343	0363	0384
02	0404	0424	0445	0465	0486	0506	0527	0547	0568	0588
03	0609	0630	0650	0671	0692	0712	0733	0754	0774	0795
04	0816	0837	0858	0878	0899	0920	0941	0962	0983	1004
05	1.1025	1046	1067	1088	1109	1130	1151	1172	1194	1215
06	1236	1257	1278	1300	1321	1342	1364	1385	1406	1428
07	1449	1470	1492	1513	1535	1556	1578	1599	1621	1642
08	1664	1686	1707	1729	1751	1772	1794	1816	1837	1859
09	1881	1903	1925	1946	1968	1990	2012	2034	2056	2078
1.10	1.2100	2122	2144	2166	2188	2210	2232	2254	2277	2299
11	2321	2343	2365	2388	2410	2432	2455	2477	2499	2522
12	2544	2566	2589	2611	2634	2656	2679	2701	2724	2746
13	2769	2792	2814	2837	2860	2882	2905	2928	2950	2973
14	2996	3019	3042	3064	3087	3110	3133	3156	3179	3202
15	1.3225	3248	3271	3294	3317	3340	3363	3386	3410	3433
16	3456	3479	3502	3526	3549	3572	3596	3619	3642	3666
17	3689	3712	3736	3759	3783	3806	3830	3853	3877	3900
18	3924	3948	3971	3995	4019	4042	4066	4090	4113	4137
19	4161	4185	4209	4232	4256	4280	4304	4328	4352	4376
1.20	1.4400	4424	4448	4472	4496	4520	4544	4568	4593	4617
21	4641	4665	4689	4714	4738	4762	4787	4811	4835	4860
22	4884	4908	4933	4957	4982	5006	5031	5055	5080	5104
23	5129	5154	5178	5203	5228	5252	5277	5302	5326	5351
24	5376	5401	5426	5450	5475	5500	5525	5550	5575	5600
25	1.5625	5650	5675	5700	5725	5750	5775	5800	5826	5851
26	5876	5901	5926	5952	5977	6002	6028	6053	6078	6104
27	6129	6154	6180	6205	6231	6256	6282	6307	6333	6358
28	6384	6410	6435	6461	6487	6512	6538	6564	6589	6615
29	6641	6667	6693	6718	6744	6770	6796	6822	6848	6874
1.30	1.6900	6926	6952	6978	7004	7030	7056	7082	7109	7135
31	7161	7187	7213	7240	7266	7292	7319	7345	7371	7398
32	7424	7450	7477	7503	7530	7556	7583	7609	7636	7662
33	7689	7716	7742	7769	7796	7822	7849	7876	7902	7929
34	7956	7983	8010	8036	8063	8090	8117	8144	8171	8198
35	1.8225	8252	8279	8306	8333	8360	8387	8414	8442	8469
36	8496	8523	8550	8578	8605	8632	8660	8687	8714	8742
37	8769	8796	8824	8851	8879	8906	8934	8961	8989	9016
38	9044	9072	9099	9127	9155	9182	9210	9238	9265	9293
39	9321	9349	9377	9404	9432	9460	9488	9516	9544	9572
1.40	1.9600	9628	9656	9684	9712	9740	9768	9796	9825	9853
41	1.9881	9909	9937	9966	9994	$\bar{0}$022	$\bar{0}$051	$\bar{0}$079	$\bar{0}$107	$\bar{0}$136
42	2.0164	0192	0221	0249	0278	0306	0335	0363	0392	0420
43	0449	0478	0506	0535	0564	0592	0621	0650	0678	0707
44	0736	0765	0794	0822	0851	0880	0909	0938	0967	0996
45	2.1025	1054	1083	1112	1141	1170	1199	1228	1258	1287
46	1316	1345	1374	1404	1433	1462	1492	1521	1550	1580
47	1609	1638	1668	1697	1727	1756	1786	1815	1845	1874
48	1904	1934	1963	1993	2023	2052	2082	2112	2141	2171
49	2201	2231	2261	2290	2320	2350	2380	2410	2440	2470
1.50	2.2500	2530	2560	2590	2620	2650	2680	2710	2741	2771
N	N² 0	1	2	3	4	5	6	7	8	9

P P	20	21	22	23	24	25	26	27	28	29	30	31
1	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0	3.1
2	4.0	4.2	4.4	4.6	4.8	5.0	5.2	5.4	5.6	5.8	6.0	6.2
3	6.0	6.3	6.6	6.9	7.2	7.5	7.8	8.1	8.4	8.7	9.0	9.3
4	8.0	8.4	8.8	9.2	9.6	10.0	10.4	10.8	11.2	11.6	12.0	12.4
5	10.0	10.5	11.0	11.5	12.0	12.5	13.0	13.5	14.0	14.5	15.0	15.5
6	12.0	12.6	13.2	13.8	14.4	15.0	15.6	16.2	16.8	17.4	18.0	18.6
7	14.0	14.7	15.4	16.1	16.8	17.5	18.2	18.9	19.6	20.3	21.0	21.7
8	16.0	16.8	17.6	18.4	19.2	20.0	20.8	21.6	22.4	23.2	24.0	24.8
9	18.0	18.9	19.8	20.7	21.6	22.5	23.4	24.3	25.2	26.1	27.0	27.9

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N².

Squares of Numbers from 1.500 to 2.000

N	N^2 0	1	2	3	4	5	6	7	8	9
1.50	2.2500	2530	2560	2590	2620	2650	2680	2710	2741	2771
51	2801	2831	2861	2892	2922	2952	2983	3013	3043	3074
52	3104	3134	3165	3195	3226	3256	3287	3317	3348	3378
53	3409	3440	3470	3501	3532	3562	3593	3624	3654	3685
54	3716	3747	3778	3808	3839	3870	3901	3932	3963	3994
55	2.4025	4056	4087	4118	4149	4180	4211	4242	4274	4305
56	4336	4367	4398	4430	4461	4492	4524	4555	4586	4618
57	4649	4680	4712	4743	4775	4806	4838	4869	4901	4932
58	4964	4996	5027	5059	5091	5122	5154	5186	5217	5249
59	5281	5313	5345	5376	5408	5440	5472	5504	5536	5568
1.60	2.5600	5632	5664	5696	5728	5760	5792	5824	5857	5889
61	5921	5953	5985	6018	6050	6082	6115	6147	6179	6212
62	6244	6276	6309	6341	6374	6406	6439	6471	6504	6536
63	6569	6602	6634	6667	6700	6732	6765	6798	6830	6863
64	6896	6929	6962	6994	7027	7060	7093	7126	7159	7192
65	2.7225	7258	7291	7324	7357	7390	7423	7456	7490	7523
66	7556	7589	7622	7656	7689	7722	7756	7789	7822	7856
67	7889	7922	7956	7989	8023	8056	8090	8123	8157	8190
68	8224	8258	8291	8325	8359	8392	8426	8460	8493	8527
69	8561	8595	8629	8662	8696	8730	8764	8798	8832	8866
1.70	2.8900	8934	8968	9002	9036	9070	9104	9138	9173	9207
71	9241	9275	9309	9344	9378	9412	9447	9481	9515	9550
72	9584	9618	9653	9687	9722	9756	9791	9825	9860	9894
73	2.9929	9964	9998	$\bar{0}033$	$\bar{0}068$	$\bar{0}102$	$\bar{0}137$	$\bar{0}172$	$\bar{0}206$	$\bar{0}241$
74	3.0276	0311	0346	0380	0415	0450	0485	0520	0555	0590
75	3.0625	0660	0695	0730	0765	0800	0835	0870	0906	0941
76	0976	1011	1046	1082	1117	1152	1188	1223	1258	1294
77	1329	1364	1400	1435	1471	1506	1542	1577	1613	1648
78	1684	1720	1755	1791	1827	1862	1898	1934	1969	2005
79	2041	2077	2113	2148	2184	2220	2256	2292	2328	2364
1.80	3.2400	2436	2472	2508	2544	2580	2616	2652	2689	2725
81	2761	2797	2833	2870	2906	2942	2979	3015	3051	3088
82	3124	3160	3197	3233	3270	3306	3343	3379	3416	3452
83	3489	3526	3562	3599	3636	3672	3709	3746	3782	3819
84	3856	3893	3930	3966	4003	4040	4077	4114	4151	4188
85	3.4225	4262	4299	4336	4373	4410	4447	4484	4522	4559
86	4596	4633	4670	4708	4745	4782	4820	4857	4894	4932
87	4969	5006	5044	5081	5119	5156	5194	5231	5269	5306
88	5344	5382	5419	5457	5495	5532	5570	5608	5645	5683
89	5721	5759	5797	5834	5872	5910	5948	5986	6024	6062
1.90	3.6100	6138	6176	6214	6252	6290	6328	6366	6405	6443
91	6481	6519	6557	6596	6634	6672	6711	6749	6787	6826
92	6864	6902	6941	6979	7018	7056	7095	7133	7172	7210
93	7249	7288	7326	7365	7404	7442	7481	7520	7558	7597
94	7636	7675	7714	7752	7791	7830	7869	7908	7947	7986
95	3.8025	8064	8103	8142	8181	8220	8259	8298	8338	8377
96	8416	8455	8494	8534	8573	8612	8652	8691	8730	8770
97	8809	8848	8888	8927	8967	9006	9046	9085	9125	9164
98	9204	9244	9283	9323	9363	9402	9442	9482	9521	9561
99	9601	9641	9681	9720	9760	9800	9840	9880	9920	9960
2.00	4.0000	0040	0080	0120	0160	0200	0240	0280	0321	0361
N	N^2 0	1	2	3	4	5	6	7	8	9

P P

	30	**31**
1	3.0	3.1
2	6.0	6.2
3	9.0	9.3
4	12.0	12.4
5	15.0	15.5
6	18.0	18.6
7	21.0	21.7
8	24.0	24.8
9	27.0	27.9

	32	**33**
1	3.2	3.3
2	6.4	6.6
3	9.6	9.9
4	12.8	13.2
5	16.0	16.5
6	19.2	19.8
7	22.4	23.1
8	25.6	26.4
9	28.8	29.7

	34	**35**
1	3.4	3.5
2	6.8	7.0
3	10.2	10,5
4	13.6	14.0
5	17.0	17.5
6	20.4	21.0
7	23.8	24.5
8	27.2	28.0
9	30.6	31.5

	36	**37**
1	3.6	3.7
2	7.2	7.4
3	10.8	11.1
4	14.4	14.8
5	18.0	18.5
6	21.6	22.2
7	25.2	25.9
8	28.8	29.6
9	32.4	33.3

	38	**39**
1	3.8	3.9
2	7.6	7.8
3	11.4	11.7
4	15.2	15.6
5	19.0	19.5
6	22.8	23.4
7	26.6	27.3
8	30.4	31.2
9	34.2	35.1

	40	**41**
1	4.0	4.1
2	8.0	8.2
3	12.0	12,3
4	16.0	16.4
5	20.0	20.5
6	24.0	24.6
7	28.0	28.7
8	32.0	32.8
9	36.0	36.9

P P

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N^2.

Squares of Numbers from 2.000 to 2.500

N	N²0	1	2	3	4	5	6	7	8	9
2.00	4.0000	0040	0080	0120	0160	0200	0240	0280	0321	0361
01	0401	0441	0481	0522	0562	0602	0643	0683	0723	0764
02	0804	0844	0885	0925	0966	1006	1047	1087	1128	1168
03	1209	1250	1290	1331	1372	1412	1453	1494	1534	1575
04	1616	1657	1698	1738	1779	1820	1861	1902	1943	1984
05	4.2025	2066	2107	2148	2189	2230	2271	2312	2354	2395
06	2436	2477	2518	2560	2601	2642	2684	2725	2766	2808
07	2849	2890	2932	2973	3015	3056	3098	3139	3181	3222
08	3264	3306	3347	3389	3431	3472	3514	3556	3597	3639
09	3681	3723	3765	3806	3848	3890	3932	3974	4016	4058
2.10	4.4100	4142	4184	4226	4268	4310	4352	4394	4437	4479
11	4521	4563	4605	4648	4690	4732	4775	4817	4859	4902
12	4944	4986	5029	5071	5114	5156	5199	5241	5284	5326
13	5369	5412	5454	5497	5540	5582	5625	5668	5710	5753
14	5796	5839	5882	5924	5967	6010	6053	6096	6139	6182
15	4.6225	6268	6311	6354	6397	6440	6483	6526	6570	6613
16	6656	6699	6742	6786	6829	6872	6916	6959	7002	7046
17	7089	7132	7176	7219	7263	7306	7350	7393	7437	7480
18	7524	7568	7611	7655	7699	7742	7786	7830	7873	7917
19	7961	8005	8049	8092	8136	8180	8224	8268	8312	8356
2.20	4.8400	8444	8488	8532	8576	8620	8664	8708	8753	8797
21	8841	8885	8929	8974	9018	9062	9107	9151	9195	9249
22	9284	9328	9373	9417	9462	9506	9551	9595	9640	9684
23	4.9729	9774	9818	9863	9908	9952	9997	$\bar{0}042$	$\bar{0}086$	$\bar{0}131$
24	5.0176	0221	0266	0310	0355	0400	0445	0490	0535	0580
25	5.0625	0670	0715	0760	0805	0850	0895	0940	0986	1031
26	1076	1121	1166	1212	1257	1302	1348	1393	1438	1484
27	1529	1574	1620	1665	1711	1756	1802	1847	1893	1938
28	1984	2030	2075	2121	2167	2212	2258	2304	2349	2395
29	2441	2487	2533	2578	2624	2670	2716	2762	2808	2854
2.30	5.2900	2946	2992	3038	3084	3130	3176	3222	3269	3315
31	3361	3407	3453	3500	3546	3592	3639	3685	3731	3778
32	3824	3870	3917	3963	4010	4056	4103	4149	4196	4242
33	4289	4336	4382	4429	4476	4522	4569	4616	4662	4709
34	4756	4803	4850	4896	4943	4990	5037	5084	5131	5178
35	5.5225	5272	5319	5366	5413	5460	5507	5554	5602	5649
36	5696	5743	5790	5838	5885	5932	5980	6027	6074	6122
37	6169	6216	6264	6311	6359	6406	6454	6501	6549	6596
38	6644	6692	6739	6787	6835	6882	6930	6978	7025	7073
39	7121	7169	7217	7264	7312	7360	7408	7456	7504	7552
2.40	5.7600	7648	7696	7744	7792	7840	7888	7936	7985	8033
41	8081	8129	8177	8226	8274	8322	8371	8419	8467	8516
42	8564	8612	8661	8709	8758	8806	8855	8903	8952	9000
43	9049	9098	9146	9195	9244	9292	9341	9390	9438	9487
44	5.9536	9585	9634	9682	9731	9780	9829	9878	9927	9976
45	6.0025	0074	0123	0172	0221	0270	0319	0368	0418	0467
46	0516	0565	0614	0664	0713	0762	0812	0861	0910	0960
47	1009	1058	1108	1157	1207	1256	1306	1355	1405	1454
48	1504	1554	1603	1653	1703	1752	1802	1852	1901	1951
49	2001	2051	2101	2150	2200	2250	2300	2350	2400	2450
2.50	6.2500	2550	2600	2650	2700	2750	2800	2850	2901	2951
N	N²0	1	2	3	4	5	6	7	8	9

P P

	40	**41**		**42**	**43**		**44**	**45**
1	4.0	4.1	1	4.2	4.3	1	4.4	4.5
2	8.0	8.2	2	8.4	8.6	2	8.8	9.0
3	12.0	12.3	3	12.6	12.9	3	13.2	13.5
4	16.0	16.4	4	16.8	17.2	4	17.6	18.0
5	20.0	20.5	5	21.0	21.5	5	22.0	22.5
6	24.0	24.6	6	25.2	25.8	6	26.4	27.0
7	28.0	28.7	7	29.4	30.1	7	30.8	31.5
8	32.0	32.8	8	33.6	34.4	8	35.2	36.0
9	36.0	36.9	9	37.8	38.7	9	39.6	40.5

	46	**47**		**48**	**49**		**50**	**51**
1	4.6	4.7	1	4.8	4.9	1	5.0	5.1
2	9.2	9.4	2	9.6	9.8	2	10.0	10.2
3	13.8	14.1	3	14.4	14.7	3	15.0	15.3
4	18.4	18.8	4	19.2	19.6	4	20.0	20.4
5	23.0	23.5	5	24.0	24.5	5	25.0	25.5
6	27.6	28.2	6	28.8	29.4	6	30.0	30.6
7	32.2	32.9	7	33.6	34.3	7	35.0	35.7
8	36.8	37.6	8	38.4	39.2	8	40.0	40.8
9	41.4	42.3	9	43.2	44.1	9	45.0	45.9

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N².

Squares of Numbers from 2.500 to 3.000

N	N²0	1	2	3	4	5	6	7	8	9
2.50	6.2500	2550	2600	2650	2700	2750	2800	2850	2901	2951
51	3001	3051	2101	3152	3202	3252	3303	3353	3403	3454
52	3504	3554	3605	3655	3706	3756	3807	3857	3908	3958
53	4009	4060	4110	4161	4212	4262	4313	4364	4414	4465
54	4516	4567	4618	4668	4719	4770	4821	4872	4923	4974
55	6.5025	5076	5127	5178	5229	5280	5331	5382	5434	5485
56	5536	5587	5638	5690	5741	5792	5844	5895	5946	5998
57	6049	6100	6152	6203	6255	6306	6358	6409	6461	6512
58	6564	6616	6667	6719	6771	6822	6874	6926	6977	7029
59	7081	7133	7185	7236	7288	7340	7392	7444	7496	7548
2.60	6.7600	7652	7704	7756	7808	7860	7912	7964	8017	8069
61	8121	8173	8225	8278	8330	8382	8435	8487	8539	8592
62	8644	8696	8749	8801	8854	8906	8959	9011	9064	9116
63	9169	9222	9274	9327	9380	9432	9485	9538	9590	9643
64	6.9696	9749	9802	9854	9907	9960	0̄013	0̄066	0̄119	0̄172
65	7.0225	0278	0331	0384	0437	0490	0543	0596	0650	0703
66	0756	0809	0862	0916	0969	1022	1076	1129	1182	1236
67	1289	1342	1396	1449	1503	1556	1610	1663	1717	1770
68	1824	1878	1931	1985	2039	2092	2146	2200	2253	2307
69	2361	2415	2469	2522	2576	2630	2684	2738	2792	2846
2.70	7.2900	2954	3008	3062	3116	3170	3224	3278	3333	3387
71	3441	3495	3549	3604	3658	3712	3767	3821	3875	3930
72	3984	4038	4093	4147	4202	4256	4311	4365	4420	4474
73	4529	4584	4638	4693	4748	4802	4857	4912	4966	5021
74	5076	5131	5186	5240	5295	5350	5405	5460	5515	5570
75	7.5625	5680	5735	5790	5845	5900	5955	6010	6066	6121
76	6176	6231	6286	6342	6397	6452	6508	6563	6618	6674
77	6729	6784	6840	6895	6951	7006	7062	7117	7173	7228
78	7284	7340	7395	7451	7507	7562	7618	7674	7729	7785
79	7841	7897	7953	8008	8064	8120	8176	8232	8288	8344
2.80	7.8400	8456	8512	8568	8624	8680	8736	8792	8849	8905
81	8961	9017	9073	9130	9186	9242	9299	9355	9411	9468
82	7.9524	9580	9637	9693	9750	9806	9863	9919	9976	0̄032
83	8.0089	0146	0202	0259	0316	0372	0429	0486	0542	0599
84	0656	0713	0770	0826	0883	0940	0997	1054	1111	1168
85	8.1225	1282	1339	1396	1453	1510	1567	1624	1682	1739
86	1796	1853	1910	1968	2025	2082	2140	2197	2254	2312
87	2369	2426	2484	2541	2599	2656	2714	2771	2829	2886
88	2944	3002	3059	3117	3175	3232	3290	3348	3405	3463
89	3521	3579	3637	3694	3752	3810	3868	3926	3984	4042
2.90	8.4100	4158	4216	4274	4332	4390	4448	4506	4565	4623
91	4681	4739	4797	4856	4914	4972	5031	5089	5147	5206
92	5264	5322	5381	5439	5498	5556	5615	5673	5732	5790
93	5849	5908	5966	6025	6084	6142	6201	6260	6318	6377
94	6436	6495	6554	6612	6671	6730	6789	6848	6907	6966
95	8.7025	7084	7143	7202	7261	7320	7379	7438	7498	7557
96	7616	7675	7734	7794	7853	7912	7972	8031	8090	8150
97	8209	8268	8328	8387	8447	8506	8566	8625	8685	8744
98	8804	8864	8923	8983	9043	9102	9162	9222	9281	9341
99	8.9401	9461	9521	9580	9640	9700	9760	9820	9880	9940
3.00	9.0000	0060	0120	0180	0240	0300	0360	0420	0481	0541
N	N²0	1	2	3	4	5	6	7	8	9

P P

p	50	51
1	5.0	5.1
2	10.0	10.2
3	15.0	15.3
4	20.0	20.4
5	25.0	25.5
6	30.0	30.6
7	35.0	35.7
8	40.0	40.8
9	45.0	45.9
	52	53
1	5.2	5.3
2	10.4	10.6
3	15.6	15.9
4	20.8	21.2
5	26.0	26.5
6	31.2	31.8
7	36.4	37 1
8	41.6	42.4
9	46.8	47.7
	54	55
1	5.4	5.5
2	10.8	11.0
3	16.2	16.5
4	21.6	22.0
5	27.0	27.5
6	32.4	33.0
7	37.8	38.5
8	43.2	44.0
9	48.6	49.5
	56	57
1	5.6	5.7
2	11.2	11.4
3	16.8	17.1
4	22.4	22.8
5	28.0	28.5
6	33.6	34.2
7	39.2	39.9
8	44.8	45.6
9	50.4	51.3
	58	59
1	5.8	5.9
2	11.6	11.8
3	17.4	17.7
4	23.2	23.6
5	29.0	29.5
6	34.8	35.4
7	40.6	41.3
8	46.4	47.2
9	52.2	53.1
	60	61
1	6.0	6.1
2	12.0	12.2
3	18.0	18.3
4	24.0	24.4
5	30.0	30.5
6	36.0	36.6
7	42.0	42.7
8	48.0	48.8
9	54.0	54.9

P P

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N².

Squares of Numbers from 3.000 to 3.500

N	N²0	1	2	3	4	5	6	7	8	9
3.00	9.0000	0060	0120	0180	0240	0300	0360	0420	0481	0541
01	0601	0661	0721	0782	0842	0902	0963	1023	1083	1144
02	1204	1264	1325	1385	1446	1506	1567	1627	1688	1748
03	1809	1870	1930	1991	2052	2112	2173	2234	2294	2355
04	2416	2477	2538	2598	2659	2720	2781	2842	2903	2964
05	9.3025	3086	3147	3208	3269	3330	3391	3452	3514	3575
06	3636	3697	3758	3820	3881	3942	4004	4065	4126	4188
07	4249	4310	4372	4433	4495	4556	4618	4679	4741	4802
08	4864	4926	4987	5049	5111	5172	5234	5296	5357	5419
09	5481	5543	5605	5666	5728	5790	5852	5914	5976	6038
3.10	9.6100	6162	6224	6286	6348	6410	6472	6534	6597	6659
11	6721	6783	6845	6908	6970	7032	7095	7157	7219	7282
12	7344	7406	7469	7531	7594	7656	7719	7781	7844	7906
13	7969	8032	8094	8157	8220	8282	8345	8408	8470	8533
14	8596	8659	8722	8784	8847	8910	8973	9036	9099	9162
15	9.9225	9288	9351	9414	9477	9540	9603	9666	9730	9793
16	9.9856	9919	9982	0̄046	0̄109	0̄172	0̄236	0̄299	0362	0̄426
17	10.0489	0552	0616	0679	0743	0806	0870	0933	0997	1060
18	1124	1188	1251	1315	1379	1442	1506	1570	1633	1697
19	1761	1825	1889	1952	2016	2080	2144	2208	2272	2336
3.20	10.2400	2464	2528	2592	2656	2720	2784	2848	2913	2977
21	3041	3105	3169	3234	3298	3362	3427	3491	3555	3620
22	3684	3748	3813	3877	3942	4006	4071	4135	4200	4264
23	4329	4394	4458	4523	4588	4652	4717	4782	4846	4911
24	4976	5041	5106	5170	5235	5300	5365	5430	5495	5560
25	10.5625	5690	5755	5820	5885	5950	6015	6080	6146	6211
26	6276	6341	6406	6472	6537	6602	6668	6733	6798	6864
27	6929	6994	7060	7125	7191	7256	7322	7387	7453	7518
28	7584	7650	7715	7781	7847	7912	7978	8044	8109	8175
29	8241	8307	8373	8438	8504	8570	8636	8702	8768	8834
3.30	10.8900	8966	9032	9098	9164	9230	9296	9362	9429	9495
31	10.9561	9627	9693	9760	9826	9892	9959	0̄025	0̄091	0̄158
32	11.0224	0290	0357	0423	0490	0556	0623	0689	0756	0822
33	0889	0956	1022	1089	1156	1222	1289	1356	1422	1489
34	1556	1623	1690	1756	1823	1890	1957	2024	2091	2158
35	11.2225	2292	2359	2426	2493	2560	2627	2694	2762	2829
36	2896	2963	3030	3098	3165	3232	3300	3367	3434	3502
37	3569	3636	3704	3771	3839	3906	3974	4041	4109	4176
38	4244	4312	4379	4447	4515	4582	4650	4718	4785	4853
39	4921	4989	5057	5124	5192	5260	5328	5396	5464	5532
3.40	11.5600	5668	5736	5804	5872	5940	6008	6076	6145	6213
41	6281	6349	6417	6486	6554	6622	6691	6759	6827	6896
42	6964	7032	7101	7169	7238	7306	7375	7443	7512	7580
43	7649	7718	7786	7855	7924	7992	8061	8130	8198	8267
44	8336	8405	8474	8542	8611	8680	8749	8818	8887	8956
45	11.9025	9094	9163	9232	9301	9370	9439	9508	9578	9647
46	11.9716	9785	9854	9924	9993	0̄062	0̄132	0̄201	0̄270	0̄340
47	12.0409	0478	0548	0617	0687	0756	0826	0895	0965	1034
48	1104	1174	1243	1313	1383	1452	1522	1592	1661	1731
49	1801	1871	1941	2010	2080	2150	2220	2290	2360	2430
3.50	12.2500	2570	2640	2710	2780	2850	2920	2990	3061	3131
N	N²0	1	2	3	4	5	6	7	8	9

P P

	60	61	62	63	64	65	66	67	68	69	70	71
1	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0	7.1
2	12.0	12.2	12.4	12.6	12.8	13.0	13.2	13.4	13.6	13.8	14.0	14.2
3	18.0	18.3	18.6	18.9	19.2	19.5	19.8	20.1	20.4	20.7	21.0	21.3
4	24.0	24.4	24.8	25.2	25.6	26.0	26.4	26.8	27.2	27.6	28.0	28.4
5	30.0	30.5	31.0	31.5	32.0	32.5	33.0	33.5	34.0	34.5	35.0	35.5
6	36.0	36.6	37.2	37.8	38.4	39.0	39.6	40.2	40.8	41.4	42.0	42.6
7	42.0	42.7	43.4	44.1	44.8	45.5	46.2	46.9	47.6	48.3	49.0	49.7
8	48.0	48.8	49.6	50.4	51.2	52.0	52.8	53.6	54.4	55.2	56.0	56.8
9	54.0	54.9	55.8	56.7	57.6	58.5	59.4	60.3	61.2	62.1	63.0	63.9

Moving the decimal point *one* place in N is equivalent to moving it *two* places in N².

TABLE VII

TABLE FOR TRANSFORMING ANGLES

To change from Minutes and Seconds into the Decimal Parts of a Degree

From Seconds		From Minutes	
1″ = 0°.00028	8″ = 0°.00222	1′ = 0°.017	8′ = 0°.133
2″ = 0°.00056	9″ = 0°.00250	2′ = 0°.033	9′ = 0°.150
3″ = 0°.00083	10″ = 0°.00278	3′ = 0°.050	10′ = 0°.167
4″ = 0°.00111	20″ = 0°.00556	4′ = 0°.067	20′ = 0°.333
5″ = 0°.00139	30″ = 0°.00833	5′ = 0°.083	30′ = 0°.500
6″ = 0°.00167	40″ = 0°.01111	6′ = 0°.100	40′ = 0°.667
7″ = 0°.00194	50″ = 0°.01389	7′ = 0°.117	50′ = 0°.833

To change from Decimal Parts of a Degree into Minutes and Seconds

0°.0000 = 0′.000 = 0″	0°.20 = 12′.0 = 12′	0°.60 = 36′.0 = 36′
0°.0001 = 0′.006 = 0″.36	0°.21 = 12′.6 = 12′ 36″	0°.61 = 36′.6 = 36′ 36″
0°.0002 = 0′.012 = 0″.72	0°.22 = 13′.2 = 13′ 12″	0°.62 = 37′.2 = 37′ 12″
0°.0003 = 0′.018 = 1″.08	0°.23 = 13′.8 = 13′ 48″	0°.63 = 37′.8 = 37′ 48″
0°.0004 = 0′.024 = 1″.44	0°.24 = 14′.4 = 14′ 24″	0°.64 = 38′.4 = 38′ 24″
0°.0005 = 0′.030 = 1″.80	0°.25 = 15′.0 = 15′	0°.65 = 39′.0 = 39′
0°.0006 = 0′.036 = 2″.16	0°.26 = 15′.6 = 15′ 36″	0°.66 = 39′.6 = 39′ 36″
0°.0007 = 0′.042 = 2″.52	0°.27 = 16′.2 = 16′ 12″	0°.67 = 40′.2 = 40′ 12″
0°.0008 = 0′.048 = 2″.88	0°.28 = 16′.8 = 16′ 48″	0°.68 = 40′.8 = 40′ 48″
0°.0009 = 0′.054 = 3″.24	0°.29 = 17′.4 = 17′ 24″	0°.69 = 41′.4 = 41′ 24″
0°.0010 = 0′.060 = 3″.60	0°.30 = 18′.0 = 18′	0°.70 = 42′.0 = 42′
0°.001 = 0′.06 = 3″.6	0°.31 = 18′.6 = 18′ 36″	0°.71 = 42′.6 = 42′ 36″
0°.002 = 0′.12 = 7″.2	0°.32 = 19′.2 = 19′ 12″	0°.72 = 43′.2 = 43′ 12″
0°.003 = 0′.18 = 10″.8	0°.33 = 19′.8 = 19′ 48″	0°.73 = 43′.8 = 43′ 48″
0°.004 = 0′.24 = 14″.4	0°.34 = 20′.4 = 20′ 24″	0°.74 = 44′.4 = 44′ 24″
0°.005 = 0′.30 = 18″.0	0°.35 = 21′.0 = 21′	0°.75 = 45′.0 = 45′
0°.006 = 0′.36 = 21″.6	0°.36 = 21′.6 = 21′ 36″	0°.76 = 45′.6 = 45′ 36″
0°.007 = 0′.42 = 25″.2	0°.37 = 22′.2 = 22′ 12″	0°.77 = 46′.2 = 46′ 12″
0°.008 = 0′.48 = 28″.8	0°.38 = 22′.8 = 22′ 48″	0°.78 = 46′.8 = 46′ 48″
0°.009 = 0′.54 = 32″.4	0°.39 = 23′.4 = 23′ 24″	0°.79 = 47′.4 = 47′ 24″
0°.010 = 0′.60 = 36″.0	0°.40 = 24′.0 = 24′	0°.80 = 48′.0 = 48′
0°.01 = 0′.6 = 36″	0°.41 = 24′.6 = 24′ 36″	0°.81 = 48′.6 = 48′ 36″
0°.02 = 1′.2 = 1′ 12″	0°.42 = 25′.2 = 25′ 12″	0°.82 = 49′.2 = 49′ 12″
0°.03 = 1′.8 = 1′ 48″	0°.43 = 25′.8 = 25′ 48″	0°.83 = 49′.8 = 49′ 48″
0°.04 = 2′.4 = 2′ 24″	0°.44 = 26′.4 = 26′ 24″	0°.84 = 50′.4 = 50′ 24″
0°.05 = 3′.0 = 3′	0°.45 = 27′.0 = 27′	0°.85 = 51′.0 = 51′
0°.06 = 3′.6 = 3′ 36″	0°.46 = 27′.6 = 27′ 36″	0°.86 = 51′.6 = 51′ 36″
0°.07 = 4′.2 = 4′ 12″	0°.47 = 28′.2 = 28′ 12″	0°.87 = 52′.2 = 52′ 12″
0°.08 = 4′.8 = 4′ 48″	0°.48 = 28′.8 = 28′ 48″	0°.88 = 52′.8 = 52′ 48″
0°.09 = 5′.4 = 5′ 24″	0°.49 = 29′.4 = 29′ 24″	0°.89 = 53′.4 = 53′ 24″
0°.10 = 6′.0 = 6′	0°.50 = 30′.0 = 30′	0°.90 = 54′.0 = 54′
0°.11 = 6′.6 = 6′ 36″	0°.51 = 30′.6 = 30′ 36″	0°.91 = 54′.6 = 54′ 36″
0°.12 = 7′.2 = 7′ 12″	0°.52 = 31′.2 = 31′ 12″	0°.92 = 55′.2 = 55′ 12″
0°.13 = 7′.8 = 7′ 48″	0°.53 = 31′.8 = 31′ 48″	0°.93 = 55′.8 = 55′ 48″
0°.14 = 8′.4 = 8′ 24″	0°.54 = 32′.4 = 32′ 24″	0°.94 = 56′.4 = 56′ 24″
0°.15 = 9′.0 = 9′	0°.55 = 33′.0 = 33′	0°.95 = 57′.0 = 57′
0°.16 = 9′.6 = 9′ 36″	0°.56 = 33′.6 = 33′ 36″	0°.96 = 57′.6 = 57′ 36″
0°.17 = 10′.2 = 10′ 12″	0°.57 = 34′.2 = 34′ 12″	0°.97 = 58′.2 = 58′ 12″
0°.18 = 10′.8 = 10′ 48″	0°.58 = 34′.8 = 34′ 48″	0°.98 = 58′.8 = 58′ 48″
0°.19 = 11′.4 = 11′ 24″	0°.59 = 35′.4 = 35′ 24″	0°.99 = 59′.4 = 59′ 24″
0°.20 = 12′.0 = 12′	0°.60 = 36′.0 = 36′	1°.00 = 60′.0 = 60′

TABLE VIII — CONSTANTS

MATHEMATICAL CONSTANTS

	Logarithm
Ratio of circumference of a circle to its diameter $\pi = 3.14159265$	0.49714987
One radian = 57°.29578	1.75812263
One radian = 3437′.74677	3.53627388
One radian = 206264″.806	5.31442513
One degree = 0.01745329 radians	8.24187737 − 10
One minute = 0.00029089 radians	6.46372612 − 10
One second = 0.00000485 radians	4.68557487 − 10
Sin 1″ = 0.00000485	4.68557487 − 10
Base of natural logarithms $e = 2.71828183$	0.43429448
Modulus of common logarithms M = 0.43429448	9.63778431 − 10

RELATION BETWEEN ENGLISH AND METRIC STANDARDS OF LENGTH

1 inch = 2.54001 centimeters, 1 centimeter = 0.393700 inches.
1 foot = 0.304801 meters, 1 meter = 3.28083 feet.
1 mile = 1.60935 kilometers, 1 kilometer = 0.62137 miles.
1 nautical mile = 6080.27 feet = 1.85325 kilometers.

GEODETIC, ASTRONOMICAL, AND PHYSICAL CONSTANTS

Equatorial semi-diameter of the Earth (Clarke),	3963.3 miles.
Polar semi-diameter of the Earth (Clarke),	3949.8 miles.
Equatorial horizontal parallax of Sun,	8″.80.
Mean distance of Sun from the Earth,	92,897,000 miles.
Mean parallax of the moon,	57′ 2″.
Mean distance of the Moon from the Earth,	238,840 miles.
Velocity of light in vacuum (Newcomb),	186,326 miles per second.
Velocity of sound in dry air at 0° centigrade,	1090 feet per second.

TABLE IX

THREE-PLACE VALUES OF THE TRIGONOMETRIC FUNCTIONS

Angle	Sin	Tan	Sec	Csc	Cot	Cos	
0°	.000	.000	1.000	∞	∞	1.000	90°
1	.017	.017	1.000	57.299	57.290	1.000	89
2	.035	.035	1.001	28.654	28.636	.999	88
3	.052	.052	1.001	19.107	19.081	.999	87
4	.070	.070	1.002	14.336	14.301	.998	86
5	.087	.087	1.004	11.474	11.430	.996	85
6	.105	.105	1.006	9.567	9.514	.995	84
7	.122	.123	1.008	8.206	8.144	.993	83
8	.139	.141	1.010	7.185	7.115	.990	82
9	.156	.158	1.012	6.392	6.314	.988	81
10	.174	.176	1.015	5.759	5.671	.985	80
11	.191	.194	1.019	5.241	5.145	.982	79
12	.208	.213	1.022	4.810	4.705	.978	78
13	.225	.231	1.026	4.445	4.331	.974	77
14	.242	.249	1.031	4.134	4.001	.970	76
15	.259	.268	1.035	3.864	3.732	.966	75
16	.276	.287	1.040	3.628	3.487	.961	74
17	.292	.306	1.046	3.420	3.271	.956	73
18	.309	.325	1.051	3.236	3.078	.951	72
19	.326	.344	1.058	3.072	2.904	.946	71
20	.342	.364	1.064	2.924	2.747	.940	70
21	.358	.384	1.071	2.790	2.605	.934	69
22	.375	.404	1.079	2.669	2.475	.927	68
23	.391	.424	1.086	2.559	2.356	.921	67
24	.407	.445	1.095	2.459	2.246	.914	66
25	.423	.466	1.103	2.366	2.145	.906	65
26	.438	.488	1.113	2.281	2.050	.899	64
27	.454	.510	1.122	2.203	1.963	.891	63
28	.469	.532	1.133	2.130	1.881	.883	62
29	.485	.554	1.143	2.063	1.804	.875	61
30	.500	.577	1.155	2.000	1.732	.866	60
31	.515	.601	1.167	1.942	1.664	.857	59
32	.530	.625	1.179	1.887	1.600	.848	58
33	.545	.649	1.192	1.836	1.540	.839	57
34	.559	.675	1.206	1.788	1.483	.829	56
35	.574	.700	1.221	1.743	1.428	.819	55
36	.588	.727	1.236	1.701	1.376	.809	54
37	.602	.754	1.252	1.662	1.327	.799	53
38	.616	.781	1.269	1.624	1.280	.788	52
39	.629	.810	1.287	1.589	1.235	.777	51
40	.643	.839	1.305	1.556	1.192	.766	50
41	.656	.869	1.325	1.524	1.150	.755	49
42	.669	.900	1.346	1.494	1.111	.743	48
43	.682	.933	1.367	1.466	1.072	.731	47
44	.695	.966	1.390	1.440	1.036	.719	46
45°	.707	1.000	1.414	1.414	1.000	.707	45°
	Cos	Cot	Csc	Sec	Tan	Sin	Angle

TABLE X

THREE-PLACE LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	519	532	544	556	568	580	591
4	602	613	623	634	644	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	887	892	898
8	903	909	914	919	924	929	935	940	945	949
9	954	959	964	969	973	978	982	987	991	996
10	000	004	009	013	017	021	025	029	033	037
11	041	045	049	053	057	061	065	068	072	076
12	079	083	086	090	093	097	100	104	107	111
13	114	117	121	124	127	130	134	137	140	143
14	146	149	152	155	158	161	164	167	170	173
15	176	179	182	185	188	190	193	196	199	201
16	204	207	210	212	215	218	220	223	225	228
17	230	233	236	238	241	243	246	248	250	253
18	255	258	260	263	265	267	270	272	274	277
19	279	281	283	286	288	290	292	295	297	299